GUNKOLOGY AND POINTILLISM: TWO MUTUALLY SUPERVENING MODELS OF THE REGION-BASED AND THE POINT-BASED THEORY OF THE INFINITE TWO-DIMENSIONAL CONTINUUM

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1 Introduction

In 2003, Arsenijević introduced the generalized concept of syntactically and semantically trivial differences between formal theories[1]. According to his definition, two formal theories that can be interpreted in no model so that the variables of one of them and the variables of the other one range over the elements of one and the same basic set should still be said to be just trivially different if there are two sets of structure preserving translation rules that map one-one, respectively, the infinite set of all the formulae of one of the theories into a set of formulae of the other one and provide, at the same time, that all the truths about the elements and their relations in any model of one of the two theories are unequivocally expressed as truths about basic elements and their relations in a model of the other theory, and vice versa. What matters in such cases is not the sameness of the elements of the basic sets in a model of two theories but the equivalence of the truth expressiveness of the two theories, that is, the fact that we can use any of the two theories to express all the truths of

[1] See Arsenijević (2003).

the other theory. In any of such cases, any model of one of the two theories can be said to supervene on the corresponding model of the other one.

The reason why in such cases the resulting mappings between the two sets of formulae must be Felix Bernstein's mappings[2] from each of the sets into and not onto another set of formulae follows from the fact that, since the variables of the two theories supposedly cannot range over the elements of one and the same basic set, each element of the basic set of a model of one of the theories must be unequivocally associated with more than one elements in the corresponding model of the other theory, and so, each formula of the language into which we translate must have more variables than the corresponding formula of the language from which we translate.

In their two articles[3], Arsenijević and Kapetanović proved that the Cantorian point-based and the Aristotelian stretch-based system of the infinite linear continuum are just trivially different in the generalized sense. Each point in a model of the point-based system can be corresponded to the abutment place of two equivalence classes of abutting stretches in the corresponding model of the stretch-based system and represented through a pair of stretches, whereas each stretch in a model of the stretch-based system can be corresponded to and represented through a pair of (end-)points in the corresponding model of the point-based system.

In the present article, we want to prove that the same result holds for the point-based and the region-based system of the infinite two-dimensional continuum, so that, contrary to those who believe that the two systems represent interesting alternatives to each other, each of them can be used equally well for expressing all the truths about its own basic elements and their relations as well about its rival's basic elements and their relations. In particular, as

[2]See Cantor (1962), p. 450.
[3]See Arsenijević and Kapetanović (2008a) and Arsenijević and Kapetanović (2008b).

we shall see, any of the two theories can be used at will for expressing truths about one-dimensional entities, which, not being the elements in any model of any of the two systems, can still be spoken of as entities supervening either on null-dimensional or on two-dimensional entities of an infinite two-dimensional continuous structure. This will be an additional illustration of why what matters is not the sameness of the set of basic elements in a model but the truth expressiveness of the systems concerning all the entities and their relations present in the model explicitly or implicitly.

2 The region-based system S_R

For two reasons we shall formulate both the region-based system S_R and the point-based system S_P in the infinitary language $L_{\omega_1\omega_1}$. Both reasons have to do with the main point of the paper. First, since the first-order language is not sufficient for the formulation of the axioms defining a continuous structure, we need some stronger language, but we also want to avoid the standard second-order language, since in S_R we want to speak explicitly only about regions and their relations, just as in S_P we want to speak explicitly only about points and their relations. So, the language $L_{\omega_1\omega_1}$ represents an appropriate and the weakest possible extension of the first-order language in which we may let the variables of S_R range exclusively over the set of regions and the variables of S_P range exclusively over the set of points as two respective universes of discourse.[4] In this way, we straightforwardly get that to speak of other entities, such as lines, will mean to speak directly just of regions and their relations or about points and their relations. The second reason for choosing $L_{\omega_1\omega_1}$ is that it will enable us to formulate the two sets of translation rules as directly related to regions as

[4]As it is done, for instance, in Humblin (1969), Hamblin (1971), Needham (1981), Burgess (1982) and Bochman (1990).

the elements of the basic set of S_R and points as the elements of the basic set of S_P without having to mention explicitly any intermediary entities.[5]

Now, though in sketching the region-based system we shall proceed in a pure Hilbertian manner[6], so that what regions and their relations are will follow, after all, from the set of all the axioms implicitly defining them, let us say in advance that in any intended model the regions are meant to be circle-like entities or any other entities topologically homeomorphic to them, which should cover an infinite two-dimensional surface without gaps.

Let S_R contain – besides logical constants, $=, \forall$ and $\exists$ – individual variables $a_1, a_2, ..., a_i, ..., b_1, b_2, ..., b_i, ..., c_1, c_2, ..., c_i, ...$(sometimes also without subscripts) that will supposedly range over an infinite set of regions as the basic set of the intended model, which will further be specified through the axioms that implicitly define various relations that hold between regions. Let the only non-logical relation symbol be $|$, which will, analogously to Scheffer's stroke, serve to define all relations that we want to hold between regions. Intuitively, $a \mid b$ says that regions a and b are connected.

For any kind of connection between any two regions, the following three axioms should hold [7], which we shall call *The Connectedness Axioms of* S_R:

(AS_R1) $\quad \forall a \, (a \mid a)$,

(AS_R2) $\quad \forall a \forall b \, (a \mid b \rightarrow b \mid a)$,

(AS_R3) $\quad \forall a \forall b \, (\forall c \, (c \mid a \leftrightarrow c \mid b) \rightarrow a = b)$.

The non-connectedness between two regions and some specific topologico - mereological kinds of the connection between two regions can be defined in the following way:

[5]Contrary to what is done in standard formalizations based on Set Theory. See, for instance, Munkres (2000), Ch. 2.

[6]See Hilbert (1902), pp. 447 ff.

[7]An axiomatisation based on the primitive relation of connection is developed in Clarke (1981).

- $a \nmid b \Leftrightarrow_{def} \neg a \mid b$, to be read as "a and b are not connected" (see *diagram 1*)

- $a \sqsubseteq b \Leftrightarrow_{def} \forall c\,(c \mid a \rightarrow c \mid b)$, to be read as "$a$ is a part of b"

- $a \sqsubset b \Leftrightarrow_{def} a \sqsubseteq b \wedge a \neq b$, to be read as "a is a proper part of b" (see *diagram 2*)

- $a \circ b \Leftrightarrow_{def} \exists c\,(c \sqsubseteq a \wedge c \sqsubseteq b)$, to be read as "$a$ and b overlap" (see *diagram 3*)

- $a \infty b \Leftrightarrow_{def} a \mid b \wedge \neg a \circ b$, to be read as "a and b are externally connected" (see *diagram 4*)

- $a \lhd b \Leftrightarrow_{def} a \sqsubset b \wedge \forall c\,(c \infty a \rightarrow \neg c \infty b)$, to be read as "$a$ is an internal part of b" (see *diagram 5*)

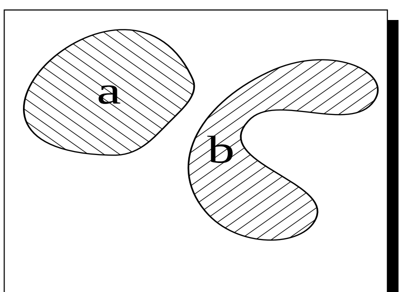

diagram 1

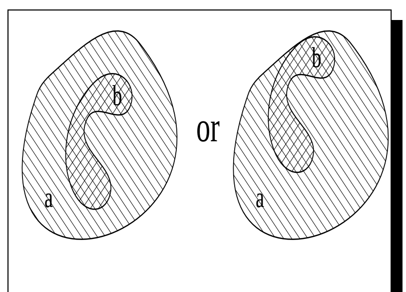

diagram 2

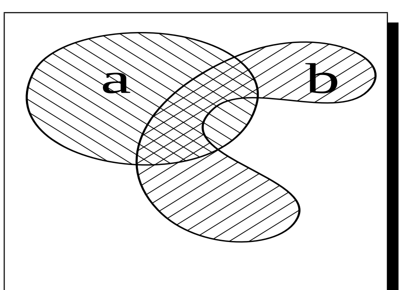

diagram 3

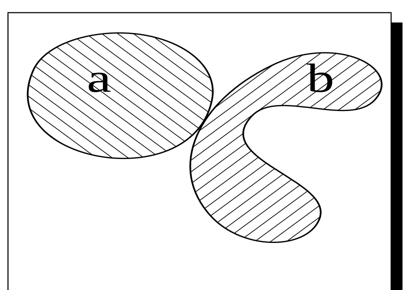

diagram 4

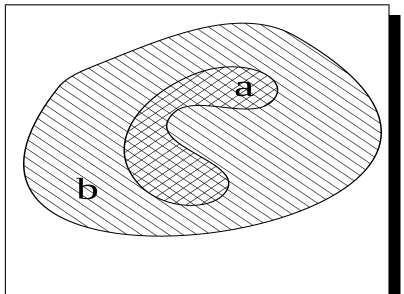

diagram 5

Notice that $\sqsubseteq$ is reflexive, antisymmetric and transitive (defining the partial ordering), whereas $\sqsubset$ and $\triangleleft$ are irreflexive, asymmetric and transitive (defining the strict ordering).

In order to preclude that two disconnected regions make up a region, we shall introduce the following axiom, which we shall call *The Non-Disconnectedness Axiom* (see *diagram 6*, where, if this were not be axiomatically precluded, a could be said to be a region consisting of a_1 and a_2):

(AS_R4) $\quad \forall a \forall b \forall c((b \triangleleft a \wedge c \triangleleft a) \rightarrow \exists d((d \mid c \wedge d \mid b) \wedge \forall e(e \triangleleft d \rightarrow e \triangleleft a)))$

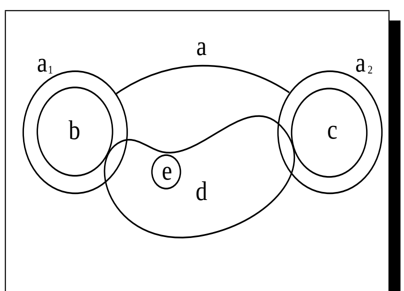

diagram 6

The following axiom, which we shall call *The Gunkness Axiom*, states that every region has proper parts, precluding the existence of atomic regions, which means that, according to the usage of "gunk" that has become popular after David introduced it[8], the continuum defined by S_R is gunky (see *diagram 7*):

[8]See Lewis (1991), pp. 20-21.

(AS_R5) $\qquad \forall a \exists b\, (b \lhd a)$

Analogously, the existence of a maximal region is precluded by the following axiom, which we shall call *The Inverse Gunkness Axiom* (see *diagram 8*):

(AS_R6) $\qquad \forall a \exists b\, (a \lhd b)$

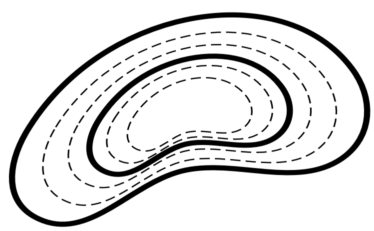

diagram 7

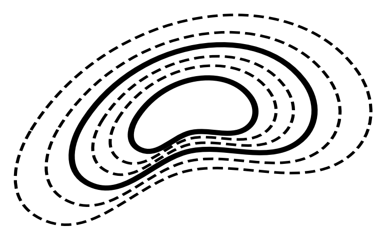

diagram 8

In order to secure that basic elements of S_R be homeomorphic to a circle and not to a two dimensional doughnut, we need the following axiom, which we shall call *The Anti-Torus Axiom* (see *diagram 9*):

(AS_R7) $\qquad \forall a \forall b (\forall c (b \lhd c \to c \circ a) \to b \triangleleft a)$

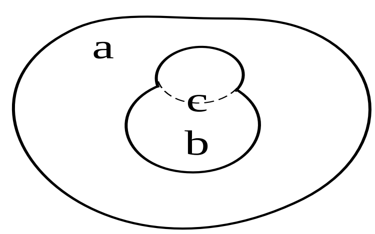

diagram 9

Let us now turn to a technically rather tricky task of defining implicitly the two-dimensionality of the intended model of S_R, which was, according to our best evidence, never done before. Namely, since we want to introduce two-dimensionality in the Hilbertian way, the set of axioms should be so selected as to be interpretable only in the structures that are two-dimensional and not more than two-dimensional. In order to obtain that, we have first, before formulating the two-dimensionality axiom, to define some additional relations in which two or more regions can stand.

To start with, we shall say that the region b is a tangential part of the region a: $b \ll a$, if and only if the following definition is fulfilled (see *diagram 10*):

- $a \ll b \Leftrightarrow_{def} a \sqsubset b \wedge \forall c\,(a \lhd c \rightarrow \neg c \sqsubseteq b)$

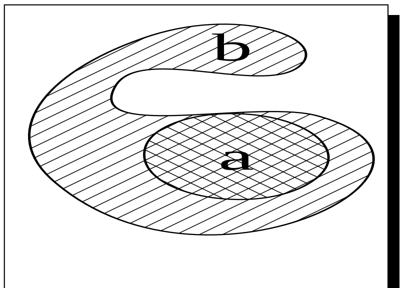

diagram 10

Now, the regions b and c will be said to be (B)ounded by the region a: $B(a, b, c)$, if and only if the following definition is fulfilled (see *diagram 11*):

$$\begin{aligned} B\,(a, b, c) \quad \Leftrightarrow_{def} \quad & b \ll a \wedge c \ll a \wedge b \infty c \wedge \neg \exists d\,((d \infty b \vee d \infty c) \wedge \\ & \wedge d \sqsubset a \wedge \forall e\,(d \sqsubseteq e \wedge e \ll a \rightarrow e \circ b \vee e \circ c) \end{aligned}$$

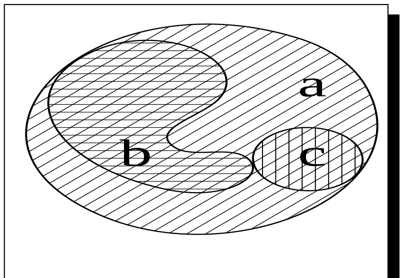

diagram 11

The regions a, b, c, and d will be said to be (N)ested: $N(a,b,c,d)$, if and only if the following definition is fulfilled (see *diagram 12*):

- $N\,(a,b,c,d) \Leftrightarrow_{def} \,(a\infty b \wedge a\infty c \wedge a\infty d \wedge b\infty c \wedge b\infty d \wedge c\infty d)$

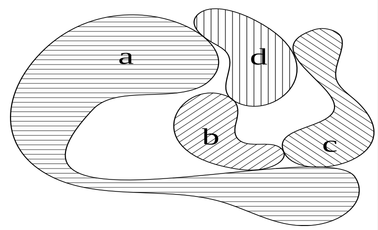

diagram 12

The regions a, b, c, and d will be said to be all mutually externally connected, (W)ittnessing the meeting at a point: $W(a,b,c,d)$, if and only if the following definition is fulfilled (see *diagram 13*):

- $W\,(a,b,c,d) \Leftrightarrow_{def} \quad \exists e\,(B\,(e,a,b) \wedge N\,(a,b,c,d) \wedge c \sqsubset e \wedge d \sqsubset e \wedge$
  $\wedge \forall f\,((f \sqsubseteq e \wedge f \mid c \wedge f \mid d) \rightarrow (f \mid a \vee f \mid b)))$

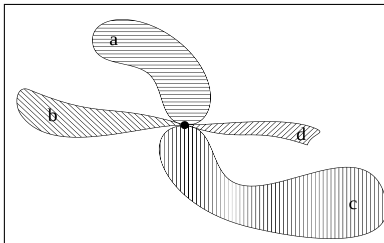

diagram 13

The two regions a and b will be said to be externally connected by meeting at a (P)oint: aPb, if and only if the following definition is fulfilled (see *diagram 14*):

- $aPb \Leftrightarrow_{def} \exists c \exists d W\,(a,b,c,d)$

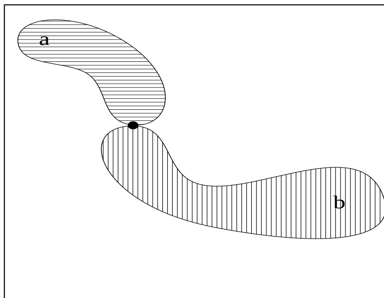

diagram 14

Here is the right place to note that we mention the *point* only informally and intuitively when we speak about regions meeting at a point since in S_R the points are not introduced explicitly. If we wanted, we could introduce them explicitly, but we do not need to do so. We may wait till they occur as elements in the point-based system and begin to speak in S_R about them only then, with the use of the translation rules holding between the two systems.

The region a will be said to meet with b and c at the point at which the regions b and c meet: $D(a, b, c)$, if and only if one of the (D)isjuncts from the following definition is true (see *diagram 15*):

- $D\,(a, b, c) \Leftrightarrow_{def} \begin{array}{l} bPc \wedge (((a \sqsubseteq b \vee b \sqsubseteq a \vee a \circ b) \wedge aPc) \vee \\ \vee((a \sqsubseteq c \vee c \sqsubseteq a \vee a \circ c) \wedge aPb)) \end{array}$

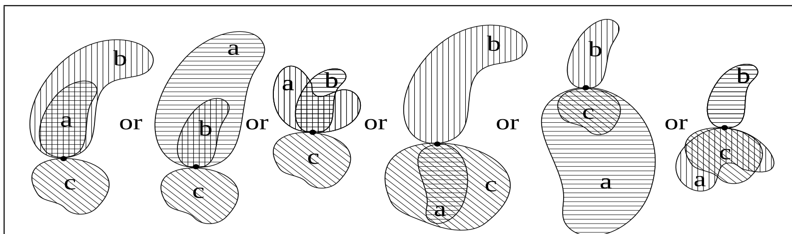

diagram 15

We are now ready to formulate the two-dimensionality axiom. First, it should be noted that the last definition says only when the region a should be said to meet with b and c at the point at which b and c meet but does not preclude the possibility that a meets b and c at the point at which b and c meet even if no disjunct from the definition of $D(a,b,c)$ is true. It is easy to see that a could meet b and c at the point at which b and c meet by approaching the meeting point along some third direction, without having any part common with c and d. But if the relational structure in which S_R is interpreted were two-dimensional, it would not be possible that no disjunct from the definition of $D(a,b,c)$ be true. So, in order to provide that the structure is two-dimensional, we have only to state, by *The Two-Dimensionality Axiom*, that one of the disjuncts is true in any given case:

$$(AS_R8) \qquad \begin{array}{l} \forall a,b,c_1,d_1,c_2,d_2(W(a,b,c_1,d_1)\wedge W(a,b,c_2,d_2)\rightarrow \\ \rightarrow D(c_1,c_2,d_1)\wedge D(c_1,c_2,d_2)) \end{array}$$

The last thing we have to do is to preclude axiomatically the possibility of "holes" within the set of all regions. This can be done analogously to the way in which Cantor precluded the existence of "gaps" in the one-dimensional continuum. The set of null-dimensional points which is dense is also continuous if any infinite accumulation of points that has a limit is such that the limit is an element of the basic set itself.[9] In our case, any infinite accumulation of regions, where a successive region contains the previous one as its internal part (see *diagram 16* below), has to be such that its limit is a region from the basic set that contains all the accumulating regions as its internal parts. The axiom will be formulated by the use of the following two definitions of $\varphi\,(a,b_1,b_2,...)$ and $\psi\,(a,b,c_1,c_2,...)$:

- $\varphi\,(a,b_1,b_2,...)\Leftrightarrow_{def} \begin{array}{l} \bigwedge_{n<\omega} b_n \lhd a \wedge \bigwedge_{m<n<\omega} b_m \lhd b_n \wedge \\ \wedge\neg\exists c(\bigwedge_{n<\omega} b_n \lhd c\wedge c\lhd a) \end{array}$

[9]cf. Cantor (1962), p. 194.

- $\psi\,(a, c, b_1, b_2, ...) \Leftrightarrow_{def} c \infty a \wedge \forall d (c \lhd d \rightarrow \bigvee_{n<\omega} b_n \circ d)$

Now, *The Continuity Axiom* reads as follows:

$(AS_R9) \qquad \forall a (\forall b_n)_{n<\omega} (\varphi(a, b_1, b_2, ...) \rightarrow \exists c \psi(a, c, b_1, b_2, ...))$

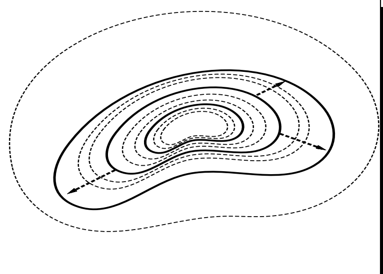

diagram 16

Obviously, the above axiom precludes the existence of a hole *outside* the region a. One may raise the question whether we should formulate an additional axiom which will preclude the existence of a hole *within* a. However, the axiom (AS_R5) together with the axiom (AS_R6) is sufficient to preclude such a possibility. Namely, according to the axiom (AS_R5), there must be a region which is a proper part of a and then, starting from *this* region, we can always apply to it the axiom (AS_R6) which guarantees the absence of the alleged holes within a.

3 The point-based system S_P

Intuitively, in order to cover completely an infinite two-dimensional plane (or a two-dimensional surface topologically homeomorphic to it) by a set of null-dimensional points, we need two sets of points, each making up an infinite set of parallels (parallel straight lines in the case of a plane or parallel quasi-straight-lines in the case of a surface topologically homeomorphic to it) such that each point from the first set of parallels is a point common with one and only one point from the other set, and vice versa. In what follows we use *parallels* to refer both to parallel lines and to quasi-parallel lines in the sense explained above (see *diagram 17*). In such a way, the continuity of each line (supervening on the set of points arranged so as to build up the linear continuum) from one of the sets will guarantee the continuous order of the lines of the other set, and vice versa.

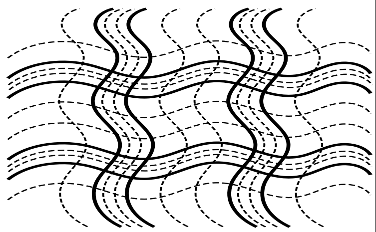

diagram 17

In order to express the required arrangement of points axiomatically, we shall first define $\varphi_1(\vec{\alpha}), ..., \varphi_{10}(\vec{\alpha})$ as shorthands for the formulae which are axioms of the points-based system of the infinite linear continuum as they are formulated in $L_{\omega_1\omega_1}$by Arsenijević and Kapetanović.[10]

1. $\varphi_1(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j<\omega}(\neg\alpha_j < \alpha_j)$

2. $\varphi_2(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j,k,l<\omega}((\alpha_j < \alpha_k \wedge \alpha_k < \alpha_l) \rightarrow \alpha_j < \alpha_l)$

3. $\varphi_3(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j,k<\omega}(\alpha_j < \alpha_k \vee \alpha_k < \alpha_j \vee \alpha_j = \alpha_k)$

4. $\varphi_4(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j,k,l<\omega}((\alpha_j = \alpha_k \wedge \alpha_j < \alpha_l) \rightarrow \alpha_k < \alpha_l)$

5. $\varphi_5(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j,k,l<\omega}((\alpha_j = \alpha_k \wedge \alpha_l < \alpha_j) \rightarrow \alpha_l < \alpha_k)$

6. $\varphi_6(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j<\omega} \alpha_j \bigvee_{k<\omega} \alpha_k(\alpha_k < \alpha_j)$

7. $\varphi_7(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j<\omega} \alpha_j \bigvee_{k<\omega} \alpha_k(\alpha_j < \alpha_k)$

[10]See Arsenijević and Kapetanović (2008a) and Arsenijević and Kapetanović (2008b).

8. $\varphi_8(\vec{\alpha}) \Leftrightarrow_{def} \bigwedge_{j,k<\omega}(\alpha_j < \alpha_k \rightarrow \bigvee_{l<\omega}(\alpha_j < \alpha_l \wedge \alpha_l < \alpha_k))$

9. $\varphi_9(\vec{\alpha}) \Leftrightarrow_{def} ((\exists\beta(\bigwedge_{j<\omega} \alpha_j < \beta) \rightarrow$
   $\rightarrow \exists\gamma(\bigwedge_{j<\omega} \alpha_j < \gamma \wedge \neg\exists\delta(\bigwedge_{j<\omega} \alpha_j < \delta \wedge \delta < \gamma)))$

10. $\varphi_{10}(\vec{\alpha}) \Leftrightarrow_{def} ((\exists\beta(\bigwedge_{j<\omega} \beta < \alpha_j) \rightarrow$
    $\rightarrow \exists\gamma(\bigwedge_{j<\omega} \gamma < \alpha_j \wedge \neg\exists\delta(\bigwedge_{j<\omega} \delta < \alpha_j \wedge \gamma < \delta)))$

It is important to note why in the above definitions we should and could omit the universal quantification present at the beginning in the corresponding axioms of the Point-Based axiomatization of the linear continuum. On the one hand, we have had to omit the universal quantification because we need the variables to be free in view of the intended definition of a set of points that make up an infinite set of parallel lines. On the other hand, in view of the way in which the above definitions are introduced, the omission of the universal quantification doesn't allow for the possibility of producing counterexamples. Let us suppose, for example, that one wants to produce a counterexample to what is implied by definition 9. For doing this, he could introduce new variables instead of those present in the definition and suppose that in this particular case there is no least upper bound in spite of the fact that there is an upper bound. However, there is nothing that can prevent us to re-introduce systematically the variables that occur in definition 9 instead of the new variables and show in this way that the alleged counterexample is just an instance of what is said in the definition.

Given the previous ten formulae, the following formula defines a set of points standing in such relations that they make up an infinite set of parallels that as such may though need not be continuously ordered:

$$\Psi\left(\vec{\alpha}^1, \vec{\alpha}^2, ...\right) \Leftrightarrow_{def} \bigwedge_{j\leq 10} \bigwedge_{i<\omega} \varphi_j\left(\vec{\alpha}^i\right) \wedge \neg\exists x, y\left(\alpha_n^i = x = y = \alpha_m^j\right),$$

for $i, j, m, n < \omega$ and $i \neq j$

Now, following the intuitive suggestion, the continuity of a set of parallels defined by the last formula will be guaranteed by letting this and some other set of parallels cut in the way intuitively described above. In addition, in order to secure that the relational structure is just two-dimensional, it is necessary not only that there are two sets of parallels that cut in the way required but also that for any third set of parallels it holds that there is no point at them that would not be one of the points from the two sets of cutting parallels. All this will be implicitly defined by the following axiom, which we may call *Descartes' Axiom* (remembering the concept of the *Cartesian Product*, as a result of which Descartes became famous in the history of mathematics[11]):

$$(AS_P1) \qquad (\exists\vec{\alpha}^n)_{n<\omega}(\exists\vec{\beta}^m)_{m<\omega}(\Psi(\vec{\alpha}^1,\vec{\alpha}^2,...) \wedge \Psi(\vec{\beta}^1,\vec{\beta}^2,...)\wedge$$
$$\wedge\alpha_j^i = \beta_k^l \wedge \neg\alpha_j^i = \beta_q^p \wedge \forall\gamma(\textstyle\bigvee_{r,s<\omega} \gamma = \alpha_s^r)),$$
$$\text{for } l \neq p, k \neq q \text{ and } i,j,l,k,p,q < \omega$$

It is important to note that, contrary to the region-based system, in which we had to preclude the possibility of "holes" by introducing a special axiom, we don't have to do that in the case of the point-based system. Namely, though no single infinite set of parallels as such precludes the existence of "holes" within it (see *diagram 18* bellow), *Descartes' Axiom* precludes the possibility that this holds for an infinite set of parallels that are continuous, since the way in which the "cut" of the two sets of parallels is defined and stated to exist by (AS_P1) imposes the continuity of all these parallels themselves (see *diagram 17* above). The existence of "holes" within a set of continuous parallels is precluded by the non-existence of Cantor "gaps" within any of the cutting parallels.

[11] See, for instance, Boyer and Merzbach (2011), p. 319.

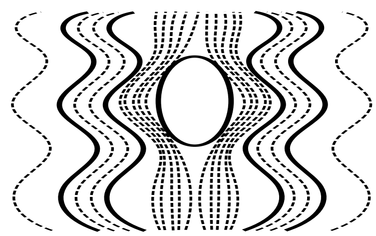

diagram 18

4 Translation rules

4.1 Translation of S_P into S_R

Intuitively, two regions in a model of S_R, a and b, which stand in the meeting-at-a-point relation aPb, meet just at one single point of the corresponding model of S_P. However, there is an infinite number of pairs of other regions that also meet at that very point. So, there is an infinite number of ways in which one and the same point of S_P can be identified within S_R. Fortunately, however, all these ways can be exhaustively classified as only 25 kinds of ways in which two pairs of regions, a and b, and c and d, can be related if a and b, and c and d should meet at one and the same point. Namely, regions a and c can stand in one of the five possible relations of the following kind: $a = c$, $a \sqsubseteq c$, $c \sqsubseteq a$, $a \circ c$, or $a \infty c$, and the same holds *mutatis mutandis* for b and d, where in the case of overlapping, say $a \circ c$, there must be a proper part of a region c which is also a proper part of a and which meets region b at-a-point, and where in the case of $a \infty c$ and $b \infty d$, the said regions must stand in the witnessing-meeting-at-a-point relation $W(a, b, c, d)$.

In order to secure that the identity of two points in S_P when spoken of in S_R is just about the unique *point* we need a function f_1, mapping the variables of S_P into the variable pairs of S_R:

$$f_1 : \alpha_n \to \langle a_{2n-1}, a_{2n} \rangle \qquad (n = 1, 2, ...)$$

In order to obtain a compact version of the first translation rule T_1, we shall introduce the shorthands for five ways in which regions a_{2m-1}, a_{2n-1}, a_{2m} and a_{2n} may be:

- $\Phi_1 (A, C) \Leftrightarrow_{def} A = C$
- $\Phi_2 (A, C) \Leftrightarrow_{def} A \sqsubseteq C$
- $\Phi_3 (A, C) \Leftrightarrow_{def} C \sqsubseteq A$
- $\Phi_4 (A, C) \Leftrightarrow_{def} A \circ C \to \begin{array}{l} \exists x, y((x \sqsubseteq A \vee x \sqsubseteq C \vee y \sqsubseteq A \vee y \sqsubseteq C) \wedge \\ \wedge (xPA \vee xPC \vee yPA \vee yPC) \end{array}$
- $\Phi_5 (A, C) \Leftrightarrow_{def} A \infty C$

where A and C stand for a_{2m-1} and a_{2n-1}, or a_{2m} and a_{2n} respectively.

The identity of α_m and α_n in S_P should be expressed by the formula of S_R that stands on the right side in the translation rule T_1:

$$(T_1) \qquad \alpha_m = \alpha_n =^{T_1} \begin{array}{l} APB \wedge CPD \wedge \bigvee_{i,j \leq 5} (\Phi_i(A, C) \wedge \Phi_j (B, D)) \wedge \\ \wedge ((\Phi_5 (A, C) \wedge \Phi_5 (B, D)) \to W (A, B, C, D)) \end{array}$$

where A, B, C and D stand for a_{2m-1}, a_{2n-1}, a_{2m} and a_{2n} respectively.

Two things here should be noted. Firstly, in defining the formula Φ_4 we could make use of variables x and y for speaking about proper parts of A and C owing to the fact that the relation defined for A and C is independent of the the way in which the the regions B and D stand (see *diagram 19*).

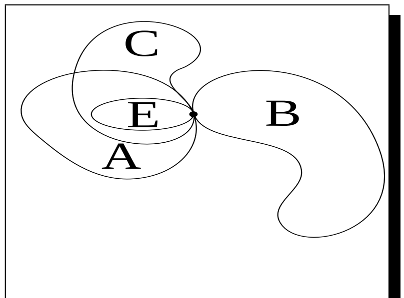

diagram 19

Secondly, we could not rely on the same technique in defining the formula Φ_5 because, unlike the previous case, we need to take into account all the four regions meeting at the same point in order to preclude that the two corresponding pairs meet at different points, and that is why we had to introduce the witnessing-meeting-at-a-point in order to preclude such a possibility. The *diagram 20* illustrates the case that should be precluded:

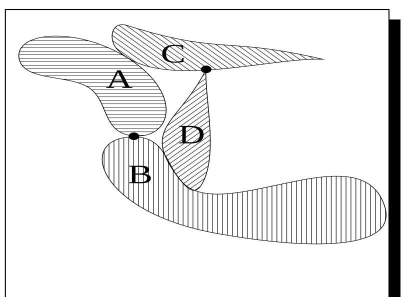

diagram 20

It is important to notice that (T_1) is *not* a *definition* that would make the formulae flanking the $=^{T_1}$ sign interchangeable within any of the two systems, since the formula on the left side of the $=^{T_1}$ sign is not a formula of S_R just as the formula on the right side is not a formula of S_P. What (T_1) enables us to do is only to *express* a truth expressed in S_P as the *corresponding* truth of S_R. "*What matters is the equivalence of the truth expressiveness and not the sameness of the sets of basic elements*".

The second translation rule should tell us how to translate the speaking of a given line within S_P into the speaking of that line within S_R. This case is particularly interesting because the lines are not basic elements neither in a model of S_P nor in a model of S_R. The lines are *supervening* entities both in the models of S_P as well in the models of S_R. So, this case shows *a fortiori* why "*what matters is the equivalence of the truth expressiveness and not the sameness of the sets of basic elements*".

In order to formulate the second translation rule, we have first to formulate both how a line is to be spoken of in S_P as well as how it is to be spoken of in S_R. The crucial difference between these formulations and the translation rule we are looking for consists in the fact that these two formations will be *definitions* that enable us to speak of lines *within* the first and the second system, respectively.

Given the definitions of $\varphi_1, \varphi_2, ..., \varphi_{10}$ above, the speaking of a line within S_P can be easily defined as speaking of the points $\chi(\alpha_1, \alpha_2, ...)$ so that:

$$\chi(\alpha_1, \alpha_2, ...) \Leftrightarrow_{def} \bigwedge_{j \leq 10} \varphi_j(\alpha_1, \alpha_2, ...)$$

For the definition of a line within S_R, we will use the definition of the tangential part $\ll$ introduced above and restrict it in two steps suggested by *diagram 21.*

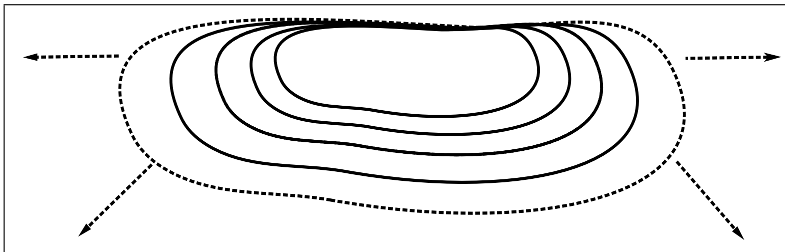

diagram 21

First, the relation $\lessdot$, as a restriction of $\ll$, will be defined as a binary relation on the set of all regions so that:

$$a \lessdot b \Leftrightarrow_{def} a \ll b \wedge \neg\forall c\,((cPa \wedge \neg c \circ b) \rightarrow cPb)$$

Intuitively, $a \lessdot b$ says that the region a is a tangential part of the region b but so that the two regions share necessarily more than a point on the boundary.

And now, secondly, let $\lessdot^*$ be the *transitive closure* of a relation $\lessdot$ i.e., the smallest transitive relation containing $\lessdot$. This means that any tangential part is developing along an infinite line.

In order to secure that by speaking about a line in S_P and in S_R we are speaking of one and the same line, we need the function f_2 mapping an infinite-tuple of variables of S_P into an infinite-tuple of variables of S_R:

$f_2 : (\alpha_1, \alpha_2, ...) \rightarrow (a_1, a_2, ...)$

So, in view of these two definitions and in view of how the speaking of a line is defined above for the system S_P, the second translation rule T_2 should read as follows:

(T_2) $\qquad \chi\,(\alpha_1, \alpha_2, ...) =^{T_2} a_1 \lessdot^* a_2 \wedge a_2 \lessdot^* a_3 \wedge ...$

4.2 Translation of S_R into S_P

Intuitively, a *region* within a model of S_P can be understood as a two-dimensional circle, or any other figure homeomorphic to it, consisting completely of a set of points. In what follows we shall refer by *one-dimensional circle* both to the genuine circle-line as well as to any closed line topologically homeomorphic to it, whereas by *two-dimensional circle* we will refer to genuine two dimensional circle-region as well as any region topologically homeomorphic to it. To find out how we can speak in S_P of an entity of the latter kind is, perhaps unexpectedly, a rather tricky task. The hint is to find a way to speak of a *line segment* and of a *closed line*, and then define a one-dimensional circle as a set of points such that it consists of all the segments having two points of the closed line as its end-points and consisting only of the elements from the set of points constituting the one-dimensional circle itself.

Following the hint, the first thing to do is to see when the points $\alpha_1, \alpha_2, ...$ make up a line segment, which we shall denote by $\varsigma(\alpha_1, \alpha_2, ...)$. Remembering the way in which a line is defined within S_P, it becomes clear that we have to leave out $\varphi_6 - \varphi_7$ from $\varphi_1 - \varphi_{10}$, which implicitly define the continuum as infinite, and introduce φ_6^* and φ_7^* instead (see *diagram 22*):

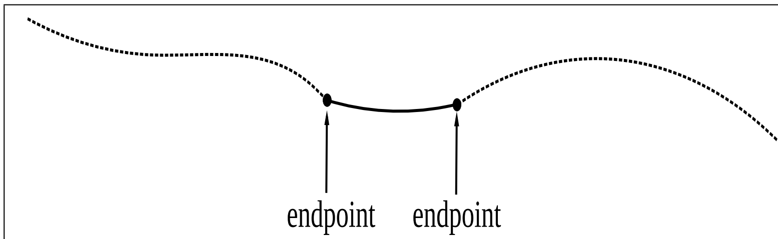

diagram 22

(φ_6^*) $\qquad \varphi_6^*(\alpha_1, \alpha_2, ...) \Leftrightarrow_{def} \bigvee_{j<\omega} \left(\bigwedge_{k<\omega} \alpha_k < \alpha_j\right)$

(φ_7^*) $\qquad \varphi_7^*(\alpha_1, \alpha_2, ...) \Leftrightarrow_{def} \bigvee_{j<\omega} \left(\bigwedge_{k<\omega} \alpha_j < \alpha_k\right)$

Now, by putting $\varphi_1 - \varphi_5$ and $\varphi_6^* - \varphi_7^*$ and $\varphi_8 - \varphi_{10}$ together, we get the following definition of a segment $\varsigma(\alpha_1, \alpha_2, ...)$:

$$\varsigma(\alpha_1, \alpha_2, ...) \Leftrightarrow_{def} \begin{array}{l} \bigwedge_{j\leq 5} \varphi_j(\alpha_1, \alpha_2, ...) \wedge \varphi_6^*(\alpha_1, \alpha_2, ...) \wedge \\ \wedge \varphi_7^*(\alpha_1, \alpha_2, ...) \wedge \bigwedge_{8\leq j\leq 10} \varphi_j(\alpha_1, \alpha_2, ...) \end{array}$$

In order to define a closed line, we need a trickier device. First, it is necessary to note that we always obtain an infinite line if we leave out a point of it (see *diagram 23*).

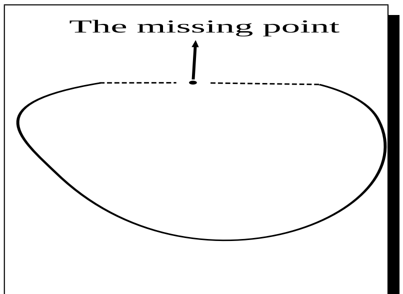

diagram 23

So, we define, first, the set of all the lines from which just one point of each of them is dropped out:

$$\pi\left(\alpha_1,\alpha_2,...\right)\Leftrightarrow_{def}\quad \begin{array}{l}\chi\left(\alpha_2,\alpha_3,...\right)\wedge\chi\left(\alpha_1,\alpha_3,...\right)\wedge ...\\ ...\wedge\chi\left(...,\alpha_{j-1},\alpha_{j+1},...\right)\wedge ...\wedge\neg\chi\left(\alpha_1,\alpha_2,...\right)\end{array}$$

and then, using this definition, we define $\tau\left(\beta_1,\beta_2,\alpha_1,\alpha_2,...\right)$ as a closed line:

$$\tau\left(\beta_1,\beta_2,\alpha_1,\alpha_2,...\right)\Leftrightarrow_{def}\quad \begin{array}{l}\varsigma\left(\alpha_1,\alpha_2,...\right)\wedge\bigvee_{i<\omega}\beta_1=\alpha_i\wedge\bigvee_{i<\omega}\beta_2=\alpha_i\wedge\\ \wedge\bigwedge_{j<\omega}\alpha_j<\beta_1\wedge\bigwedge_{j<\omega}\beta_2<\alpha_j\end{array}$$

Then, finally, we define a "full two-dimensional circle" (see *diagram 24*) as:

$$\mu\left(\alpha_1,\alpha_2,...\right)\Leftrightarrow_{def}\quad \begin{array}{l}\left(\exists\beta_i\right)_{i<\omega}\left(\exists\gamma_j\right)_{j<\omega}\left(\varsigma\left(\beta_1,\beta_2,...\right)\wedge\pi\left(\gamma_1,\gamma_2,...\right)\wedge\right.\\ \wedge\bigvee_{k,l<\omega}\tau\left(\gamma_k,\gamma_l,\beta_1,\beta_2,...\right)\wedge\\ \wedge\left(\bigvee_{m,n<\omega}\alpha_m=\beta_n\vee\bigvee_{m,n<\omega}\alpha_m=\gamma_n\right)\end{array}$$

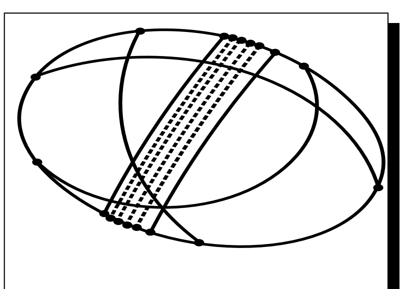

diagram 24

In order to secure that the identity of two regions in S_R when spoken of in S_P is just about the unique "region" we need a function f^*, mapping the variables of S_R into an infinite-tuple of variables of S_P:

$f_1^* : a_n \to (\alpha_{n_1}, \alpha_{n_2}, ...) \qquad (n = 1, 2, ...)$

Now, in view of all this, the two regions a_m and a_n identical in S_R should be spoken of in S_P in accordance with the following translation rule:

$$(T_1^*) \qquad a_m = a_n =^{T_1^*}$$
$$(\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...)) \to \bigwedge_{i<\omega} \alpha_{m_i} = \alpha_{n_i}$$

In order to secure that by speaking about a line in S_R and in S_P we are speaking of one and the same line, we need the function f_2^* mapping an infinite-tuple of variables of S_R into an infinite-tuple of variables of S_P:

$f_2^* : (a_1, a_2, ...) \to (\alpha_1, \alpha_2, ...)$

The next translation rule concerns speaking of the line within S_P into the speaking of the corresponding line of S_R. It would be just the inverse of the translation rule (T_2) above:

$$(T_2^*) \qquad a_1 \lessdot^* a_2 \wedge a_2 \lessdot^* a_3 \wedge ... =^{T_2^*} \chi(\alpha_1, \alpha_2, ...)$$

And now, given that all the relations in S_R are defined via $|$, the only remaining translation rule that we need is the rule concerning the connection of two regions. So, given the function f_1^*, the translation rule (T_3^*) reads as follows:

$$(T_3^*) \qquad a_m \mid a_n =^{T_3^*} \mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge \bigvee_{i,j<\omega} \alpha_{m_i} = \alpha_{n_j}$$

5 Proof that S_R and S_P are only trivially different in the generalized sense

5.1 Proof that all the axioms of S_R are theorems of S_P

Translation of (AS_R1)

According to $(T_1^*) - (T_3^*)$, the axiom

(AS_R1) $\quad \forall a\,(a \mid a)$

reads in S_P as follows:

$$(\forall \alpha_n)_{n<\omega}\,(\mu\,(\alpha_1, \alpha_2, ...) \to (\mu\,(\alpha_1, \alpha_2, ...) \wedge \mu\,(\alpha_1, \alpha_2, ...) \wedge \bigvee_{i,j<\omega} \alpha_i = \alpha_j))$$

that is, after the double application of the contraction rule:

$$(\forall \alpha_n)_{n<\omega}\,(\mu\,(\alpha_1, \alpha_2, ...) \to \bigvee_{i,j<\omega} \alpha_i = \alpha_j)$$

The last formula is trivially true in S_P, since every two dimensional circle indeed has at least one point in common with one point from the set of points from which it consists.

Translation of (AS_R2)

According to $(T_1^*) - (T_3^*)$ the axiom

(AS_R2) $\quad \forall a \forall b\,(a \mid b \to b \mid a),$

reads in S_P as follows:

$$(\forall \alpha_{m_i})_{i<\omega}(\forall \alpha_{n_j})_{j<\omega}((\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge \bigvee_{i,j<\omega} \alpha_{m_i} = \alpha_{n_j} \to$$
$$\to (\mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge \mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \bigvee_{i,j<\omega} \alpha_{n_i} = \alpha_{m_j})$$

which gives:

$$(\forall \alpha_{m_i})_{i<\omega}(\forall \alpha_{n_j})_{j<\omega}((\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge \bigvee_{i,j<\omega} \alpha_{m_i} = \alpha_{n_j}) \to$$
$$\to \bigvee_{i,j<\omega} \alpha_{n_i} = \alpha_{m_j})$$

The last formula is also trivially true in S_P, since two identical, overlapping or touching two-dimensional circles have at least one common point.

Translation of (AS_R3)

According to $(T_1^*) - (T_3^*)$ the axiom

$(AS_R3) \qquad \forall a \forall b\, (\forall c\, (c \mid a \leftrightarrow c \mid b) \rightarrow a = b).$

reads in S_P as follows:

$$(\forall \alpha_{m_i})_{i<\omega}(\forall \alpha_{n_j})_{j<\omega}(\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \rightarrow$$
$$\rightarrow (\forall \alpha_{k_l})_{l<\omega}(\mu(\alpha_{k_1}, \alpha_{k_2}, ...) \rightarrow$$
$$\rightarrow (\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{k_1}, \alpha_{k_2}, ...) \wedge \bigvee\nolimits_{i,l<\omega} \alpha_{m_i} = \alpha_{k_l}) \leftrightarrow$$
$$\leftrightarrow (\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{k_1}, \alpha_{k_2}, ...) \wedge \bigvee\nolimits_{i,l<\omega} \alpha_{m_i} = \alpha_{k_l}) \rightarrow$$
$$\rightarrow (\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \rightarrow \bigwedge\nolimits_{i<\omega} \alpha_{m_i} = \alpha_{n_i}))$$

which gives:

$$(\forall \alpha_{m_i})_{i<\omega}(\forall \alpha_{n_j})_{j<\omega}(\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \rightarrow (\forall \alpha_{k_l})_{l<\omega}$$
$$(\mu(\alpha_{k_1}, \alpha_{k_2}, ...) \rightarrow (\bigvee \alpha_{m_i} = \alpha_{k_l} \leftrightarrow \bigvee\nolimits_{i,l<\omega} \alpha_{m_i} = \alpha_{k_l}) \rightarrow$$
$$\rightarrow \bigwedge\nolimits_{i<\omega} \alpha_{m_i} = \alpha_{n_i}$$

The last formula is obviously true in S_P, for if it holds for two two-dimensional circles that every two-dimensional circle that has at least one point common to one of them has also at least one point common with the other one, the two two-dimensional circles must be identical to each other.

Translation of (AS_R4)

In order to see what we get, by translating the axiom

$(AS_R4) \qquad \forall a \forall b \forall c((b \lhd a \wedge c \lhd a) \rightarrow \exists d((d \mid c \wedge d \mid b) \wedge \forall e(e \lhd d \rightarrow e \lhd a)))$

into S_P, let us first see how the relations $\sqsubseteq$, $\sqsubset$, $\circ$, ∞ and $\lhd$ of the system S_R look like in S_P:

- $(\alpha_{m_1}, \alpha_{m_2}, ...) \sqsubseteq_p (\alpha_{n_1}, \alpha_{n_2}, ...) \Leftrightarrow_{def} \mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge$
  $\wedge (\forall \alpha_{k_l})_{l<\omega} \left(\mu(\alpha_{k_1}, \alpha_{k_2}, ...) \to \left(\bigvee \alpha_{m_i} = \alpha_{k_l} \to \bigvee_{i,l<\omega} \alpha_{m_i} = \alpha_{k_l} \right) \right)$

- $(\alpha_{m_1}, \alpha_{m_2}, ...) \sqsubset_p (\alpha_{n_1}, \alpha_{n_2}, ...) \Leftrightarrow_{def} (\alpha_{m_1}, \alpha_{m_2}, ...) \sqsubseteq_p (\alpha_{n_1}, \alpha_{n_2}, ...) \wedge$
  $\wedge \neg \bigwedge_{i<\omega} \alpha_{m_i} = \alpha_{n_i}$

- $(\alpha_{m_1}, \alpha_{m_2}, ...) \circ_p (\alpha_{n_1}, \alpha_{n_2}, ...) \Leftrightarrow_{def}$
  $\Leftrightarrow_{def} (\exists \alpha_{k_l})_{l<\omega} (\mu(\alpha_{k_1}, \alpha_{k_2}, ...) \wedge (\alpha_{k_1}, \alpha_{k_2}, ...) \sqsubseteq_p$
  $(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge (\alpha_{k_1}, \alpha_{k_2}, ...) \sqsubseteq_p (\alpha_{n_1}, \alpha_{n_2}, ...))$

- $(\alpha_{m_1}, \alpha_{m_2}, ...) \infty_p (\alpha_{n_1}, \alpha_{n_2}, ...) \Leftrightarrow_{def} \mu(\alpha_{m_1}, \alpha_{m_2}, ...) \wedge \mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge$
  $\wedge \bigvee_{i,j<\omega} \alpha_{m_i} = \alpha_{n_j} \wedge \neg (\alpha_{m_1}, \alpha_{m_2}, ...) \circ_p (\alpha_{n_1}, \alpha_{n_2}, ...)$

- $(\alpha_{m_1}, \alpha_{m_2}, ...) \lhd_p (\alpha_{n_1}, \alpha_{n_2}, ...) \Leftrightarrow_{def}$
  $\Leftrightarrow_{def} (\alpha_{m_1}, \alpha_{m_2}, ...) \sqsubset_p (\alpha_{n_1}, \alpha_{n_2}, ...) \wedge$
  $(\forall \alpha_{k_l})_{l<\omega} ((\alpha_{k_1}, \alpha_{k_2}, ...) \infty_p (\alpha_{m_1}, \alpha_{m_2}, ...) \to$
  $\to \neg (\alpha_{k_1}, \alpha_{k_2}, ...) \infty_p (\alpha_{n_1}, \alpha_{n_2}, ...))$

In view of this, the axiom (AS_R4) reads in S_P as follows:

$$(\forall \alpha_{m_i})_{i<\omega} (\forall \alpha_{n_j})_{j<\omega} (\forall \alpha_{k_l})_{l<\omega} ((\alpha_{n_1}, \alpha_{n_2}, ...) \lhd_p (\alpha_{m_1}, \alpha_{m_2}, ...) \wedge$$
$$\wedge (\alpha_{k_1}, \alpha_{k_2}, ...) \lhd_p (\alpha_{m_1}, \alpha_{m_2}, ...) \to$$
$$\to (\exists \alpha_{r_e})_{e<\omega} (\mu(\alpha_{r_1}, \alpha_{r_2}, ...) \wedge \bigvee_{l,e<\omega} \alpha_{r_e} = \alpha_{k_l} \wedge$$
$$\wedge \bigvee_{j,e<\omega} \alpha_{r_e} = \alpha_{n_j} \wedge (\forall \alpha_{s_f})_{f<\omega} ((\alpha_{s_1}, \alpha_{s_2}, ...) \lhd_p (\alpha_{r_1}, \alpha_{r_2}, ...) \to$$
$$\to (\alpha_{s_1}, \alpha_{s_2}, ...) \lhd_p (\alpha_{m_1}, \alpha_{m_2}, ...))$$

The last formula is true in S_P because, according to the definition of μ, what is required is that any two end-points of the region that consists of two separate two-dimensional circles can be connected by a line segment that contains only the points from the set of points of which the two two-dimensional circles consists, which is impossible.

Translation of (AS_R5)

According to $(T_1^*) - (T_3^*)$ and the way in which $\lhd_p$ is to be understood in S_P, the axiom

$(AS_R5) \qquad \forall a \exists b\, (b \lhd a)$

reads as follows:

$$(\forall \alpha_{m_i})_{i<\omega}(\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \rightarrow$$
$$\rightarrow (\exists \alpha_{n_j})_{j<\omega}(\mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge ((\alpha_{n_1}, \alpha_{n_2}, ...) \lhd_p (\alpha_{m_1}, \alpha_{m_2}, ...))))$$

This formula is trivially true in S_P, since it holds for every two-dimensional circle that there is a two-dimensional circle that consists of its points only but does not contain all of them.

Translation of (AS_R6)

According to $(T_1^*) - (T_3^*)$ and the way in which $\lhd_p$ is to be understood in S_P, the axiom

$(AS_R6) \qquad \forall a \exists b\, (a \lhd b)$

reads as follows:

$$(\forall \alpha_{m_i})_{i<\omega}(\mu(\alpha_{m_1}, \alpha_{m_2}, ...) \rightarrow$$
$$\rightarrow (\exists \alpha_{n_j})_{j<\omega}(\mu(\alpha_{n_1}, \alpha_{n_2}, ...) \wedge ((\alpha_{m_1}, \alpha_{m_2}, ...) \lhd_p (\alpha_{n_1}, \alpha_{n_2}, ...))))$$

The last formula is true in S_P, since no two-dimensional circle covers the whole infinite two-dimensional surface, so that it holds for every two-dimensional circle that there is a two-dimensional circle that contains all but also more points than it.

Translation of (AS_R7)

According to $(T_1^*) - (T_3^*)$ and the way in which $\lhd_p$ and $\sqsubseteq_p$ are to be understood in S_P, the axiom

$(AS_R7) \qquad \forall a \forall b(\forall c(b \lhd c \to c \circ a) \to b \lhd a)$

reads as follows

$$(\forall \alpha_{m_i})_{i<\omega}(\forall \alpha_{n_j})_{j<\omega}((\forall \alpha_{k_l})_{l<\omega}((\alpha_{n_1}, \alpha_{n_2}, ...) \lhd_p (\alpha_{k_1}, \alpha_{k_2}, ...) \to$$
$$\to (\alpha_{k_1}, \alpha_{k_2}, ...) \circ_p (\alpha_{m_1}, \alpha_{m_2}, ...)) \to (\alpha_{n_1}, \alpha_{n_2}, ...) \lhd_p (\alpha_{m_1}, \alpha_{m_2}, ...))$$

Given the way in which the regions are translated as sets of points of S_P, the last formula says nothing else but that there is no two-dimensional circle within which there could be a set of points that would not be a set of points of the two-dimensional circle itself.

Translation of (AS_R8)

In order to see what we get, by translating the axiom

$$(AS_R8) \qquad \begin{array}{l} \forall a, b, c_1, d_1, c_2, d_2(W(a, b, c_1, d_1) \wedge W(a, b, c_2, d_2) \to \\ \to D(c_1, c_2, d_1) \wedge D(c_1, c_2, d_2)) \end{array}$$

into S_P, let us first see how the relations $\ll$, B, N, W, P and D of the system S_R look like in S_P:

- $(\alpha_{m_1}, \alpha_{m_2}, ...) \ll_p (\alpha_{n_1}, \alpha_{n_2}, ...) \Leftrightarrow_{def} ((\alpha_{m_1}, \alpha_{m_2}, ...) \sqsubset_p (\alpha_{n_1}, \alpha_{n_2}, ...) \wedge$
  $\wedge (\forall \alpha_{k_l})_{l<\omega}((\alpha_{m_1}, \alpha_{m_2}, ...) \lhd_p (\alpha_{k_1}, \alpha_{k_2}, ...) \to$
  $\to \neg(\alpha_{k_1}, \alpha_{k_2}, ...) \sqsubseteq_p (\alpha_{n_1}, \alpha_{n_2}, ...))$

- $B_p \left(\alpha_{m_i}, \alpha_{n_j}, \alpha_{k_l}\right)_{i,j,l<\omega} \Leftrightarrow_{def} (\alpha_{n_1}, \alpha_{n_2}, ...) \ll_p (\alpha_{m_1}, \alpha_{m_2}, ...) \wedge$
  $\wedge (\alpha_{k_1}, \alpha_{k_2}, ...) \ll_p (\alpha_{m_1}, \alpha_{m_2}, ...) \wedge (\alpha_{n_1}, \alpha_{n_2}, ...) \infty_p (\alpha_{k_1}, \alpha_{k_2}, ...) \wedge$
  $\wedge \neg (\exists \alpha_{r_e})_{e<\omega}(((\alpha_{r_1}, \alpha_{r_2}, ...) \infty_p (\alpha_{n_1}, \alpha_{n_2}, ...) \vee$
  $\vee (\alpha_{r_1}, \alpha_{r_2}, ...) \infty_p (\alpha_{k_1}, \alpha_{k_2}, ...) \wedge$
  $\wedge (\alpha_{r_1}, \alpha_{r_2}, ...) \sqsubset_p (\alpha_{m_1}, \alpha_{m_2}, ...) \wedge$
  $\wedge \left(\forall \alpha_{s_f}\right)_{f<\omega} ((\alpha_{r_1}, \alpha_{r_2}, ...) \sqsubseteq_p (\alpha_{s_1}, \alpha_{s_2}, ...) \wedge$
  $\wedge (\alpha_{s_1}, \alpha_{s_2}, ...) \ll_p (\alpha_{m_1}, \alpha_{m_2}, ...) \to$
  $\to (\alpha_{s_1}, \alpha_{s_2}, ...) \circ_p (\alpha_{n_1}, \alpha_{n_2}, ...) \vee (\alpha_{s_1}, \alpha_{s_2}, ...) \circ_p (\alpha_{k_1}, \alpha_{k_2}, ...)))$

- $N_p\left(\alpha_{m_i},\alpha_{n_j},\alpha_{k_l},\alpha_{r_e}\right)_{i,j,l,e<\omega}\Leftrightarrow_{def}\left(\alpha_{m_1},\alpha_{m_2},...\right)\infty_p\left(\alpha_{n_1},\alpha_{n_2},...\right)\wedge$
$\wedge\left(\alpha_{m_1},\alpha_{m_2},...\right)\infty_p\left(\alpha_{r_1},\alpha_{r_2},...\right)\wedge\left(\alpha_{m_1},\alpha_{m_2},...\right)\infty_p\left(\alpha_{s_1},\alpha_{s_2},...\right)\wedge$
$\wedge\left(\alpha_{n_1},\alpha_{n_2},...\right)\infty_p\left(\alpha_{r_1},\alpha_{r_2},...\right)\wedge\left(\alpha_{n_1},\alpha_{n_2},...\right)\infty_p\left(\alpha_{s_1},\alpha_{s_2},...\right)\wedge$
$\wedge\left(\alpha_{r_1},\alpha_{r_2},...\right)\infty_p\left(\alpha_{s_1},\alpha_{s_2},...\right)$

- $W_p\left(\alpha_{m_i},\alpha_{n_j},\alpha_{k_l},\alpha_{r_e}\right)_{i,j,l,e<\omega}\Leftrightarrow_{def}$
$\Leftrightarrow_{def}\left(\exists\alpha_{s_f}\right)_{f<\omega}\left(B_p\left(\left(\alpha_{s_1},\alpha_{s_2},...\right),\left(\alpha_{m_1},\alpha_{m_2},...\right),\left(\alpha_{n_1},\alpha_{n_2},...\right)\right)\wedge\right.$
$\wedge N_p\left(\left(\alpha_{m_1},\alpha_{m_2},...\right),\left(\alpha_{n_1},\alpha_{n_2},...\right),\left(\alpha_{k_1},\alpha_{k_2},...\right),\left(\alpha_{r_1},\alpha_{r_2},...\right)\right)\wedge$
$\wedge\left(\alpha_{k_1},\alpha_{k_2},...\right)\sqsubset_p\left(\alpha_{s_1},\alpha_{s_2},...\right)\wedge\left(\alpha_{r_1},\alpha_{r_2},...\right)\sqsubset_p\left(\alpha_{s_1},\alpha_{s_2},...\right)\wedge$
$\wedge(\forall\alpha_{v_o})_{o<\omega}((\alpha_{v_1},\alpha_{v_2},...)\sqsubseteq_p\left(\alpha_{s_1},\alpha_{s_2},...\right)\wedge$
$\wedge(\mu(\alpha_{v_1},\alpha_{v_2},...)\wedge\mu(\alpha_{k_1},\alpha_{k_2},...)\wedge\bigvee_{l,o<\omega}\alpha_{v_o}=\alpha_{k_l}\wedge$
$\wedge\Big(\left(\mu\left(\alpha_{v_1},\alpha_{v_2},...\right)\wedge\mu\left(\alpha_{r_1},\alpha_{r_2},...\right)\right)\wedge\bigvee_{s,o<\omega}\alpha_{v_o}=\alpha_{r_s}\Big)\rightarrow$
$\rightarrow\Big(\left(\mu\left(\alpha_{v_1},\alpha_{v_2},...\right)\wedge\mu\left(\alpha_{m_1},\alpha_{m_2},...\right)\right)\wedge\bigvee_{i,o<\omega}\alpha_{v_o}=\alpha_{m_i}\Big)$

- $\left(\alpha_{m_1},\alpha_{m_2},...\right)P_p\left(\alpha_{n_1},\alpha_{n_2},...\right)\Leftrightarrow_{def}\left(\exists\alpha_{k_l}\right)_{l<\omega}\left(\exists\alpha_{r_e}\right)_{e<\omega}$
$W_p(\left(\alpha_{m_1},\alpha_{m_2},...\right),\left(\alpha_{n_1},\alpha_{n_2},...\right),\left(\alpha_{k_1},\alpha_{k_2},...\right),\left(\alpha_{r_1},\alpha_{r_2},...\right))$

- $D_p\left(\left(\alpha_{m_1},\alpha_{m_2},...\right),\left(\alpha_{n_1},\alpha_{n_2},...\right),\left(\alpha_{k_1},\alpha_{k_2},...\right)\right)\Leftrightarrow_{def}$
$\Leftrightarrow_{def}\left(\alpha_{m_1},\alpha_{m_2},...\right)\sqsubseteq_p\left(\alpha_{n_1},\alpha_{n_2},...\right)\vee\left(\alpha_{n_1},\alpha_{n_2},...\right)\sqsubseteq_p\left(\alpha_{m_1},\alpha_{m_2},...\right)\vee$
$\vee\left(\alpha_{m_1},\alpha_{m_2},...\right)\circ_p\left(\alpha_{n_1},\alpha_{n_2},...\right)\vee\left(\alpha_{m_1},\alpha_{m_2},...\right)\sqsubseteq_p\left(\alpha_{k_1},\alpha_{k_2},...\right)\vee$
$\vee\left(\alpha_{k_1},\alpha_{k_2},...\right)\sqsubseteq_p\left(\alpha_{m_1},\alpha_{m_2},...\right)\vee\left(\alpha_{m_1},\alpha_{m_2},...\right)\circ_p\left(\alpha_{k_1},\alpha_{k_2},...\right)$

In view of all this, the axiom (AS_R8) reads in S_P as follows

$$\left(\forall\alpha_{m_i}\right)_{i<\omega}\left(\forall\alpha_{n_j}\right)_{j<\omega}\left(\forall\alpha_{k_l}\right)_{l<\omega}\left(\forall\alpha_{r_e}\right)_{e<\omega}\left(\forall\alpha_{s_f}\right)_{f<\omega}\left(\forall\alpha_{v_o}\right)_{o<\omega}$$
$$(W_p\left(\alpha_{m_i},\alpha_{n_j},\alpha_{k_l},\alpha_{r_e}\right)_{i,j,l,e<\omega}\wedge W_p\left(\alpha_{m_i},\alpha_{n_j},\alpha_{s_f},\alpha_{v_o}\right)_{i,j,f,o<\omega}\rightarrow$$
$$\rightarrow D_p\left(\left(\alpha_{k_1},\alpha_{k_2},...\right),\left(\alpha_{s_1},\alpha_{s_2},...\right),\left(\alpha_{r_1},\alpha_{r_2},...\right)\right)\wedge$$
$$\wedge D_p\left(\left(\alpha_{k_1},\alpha_{k_2},...\right),\left(\alpha_{s_1},\alpha_{s_2},...\right),\left(\alpha_{v_1},\alpha_{v_2},...\right)\right))$$

In order to see that the last formula must be true in S_P, it is sufficient to realize that, according to (AS_P1), there can be no lines that are skew.

Translation of (AS_R9)

In order to see what we get, by translating the axiom

(AS_R9) $\quad \forall a \left(\forall b_n\right)_{n<\omega} \left(\varphi\left(a, b_1, b_2, ...\right) \rightarrow \exists c \psi\left(a, c, b_1, b_2, ...\right)\right)$

into S_P, let us first see how the following two formulae

- $\varphi\left(a, b_1, b_2, ...\right) \Leftrightarrow_{def}$

  $\Leftrightarrow_{def} \bigwedge_{n<\omega} b_n \lhd a \wedge \bigwedge_{m<n<\omega} b_m \lhd b_n \wedge \neg\exists c \left(\bigwedge_{n<\omega} b_n \lhd c \wedge c \lhd a\right)$
- $\psi\left(a, b, c_1, c_2, ...\right) \Leftrightarrow_{def} b \infty a \wedge \forall d \left(b \lhd d \rightarrow \bigvee_{n<\omega} c_n \mid d\right)$

of the system S_R look like in S_P. Recall that the symbol $\overrightarrow{\alpha}$ is used to denote a countable sequence of variables $\alpha_1, \alpha_2, ...$, with or without a subscript.

- $\varphi_p\left(\overrightarrow{\alpha}, \overrightarrow{\beta_1}, \overrightarrow{\beta_2}, ...\right) \Leftrightarrow_{def}$

  $\Leftrightarrow_{def} \mu\left(\overrightarrow{\alpha}\right) \wedge \bigwedge_{n<\omega} \mu\left(\overrightarrow{\beta_n}\right) \wedge \bigwedge_{n<\omega} \overrightarrow{\beta_n} \lhd_p \overrightarrow{\alpha} \bigwedge_{m<n<\omega} \overrightarrow{\beta_m} \lhd \overrightarrow{\beta_n} \wedge$

  $\wedge \neg\exists \overrightarrow{\gamma} (\mu\left(\overrightarrow{\gamma}\right) \wedge \bigwedge_{n<\omega} \overrightarrow{\beta_n} \lhd_p \overrightarrow{\gamma} \wedge \overrightarrow{\gamma} \lhd_p \overrightarrow{\alpha})$

- $\psi_p\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{\gamma_1}, \overrightarrow{\gamma_2}, ...\right) \Leftrightarrow_{def}$

  $\Leftrightarrow_{def} \mu\left(\overrightarrow{\alpha}\right) \wedge \mu\left(\overrightarrow{\beta}\right) \wedge \bigwedge_{n<\omega} \mu\left(\overrightarrow{\gamma_n}\right) \wedge \overrightarrow{\beta} \infty_p \overrightarrow{\alpha} \wedge$

  $\forall \overrightarrow{\delta} \left(\mu\left(\overrightarrow{\delta}\right) \wedge \overrightarrow{\beta} \lhd_p \overrightarrow{\delta} \rightarrow \bigvee_{n<\omega} \overrightarrow{\gamma_n} \mid_p \overrightarrow{\delta}\right)$

In view of all this, the axiom (AS_R9) reads in S_P as follows:

$$\forall \overrightarrow{\alpha} \left(\forall \overrightarrow{\beta_n}\right)_{n<\omega} (\mu\left(\overrightarrow{\alpha}\right) \wedge \bigwedge_{n<\omega} \mu\left(\overrightarrow{\beta_n}\right) \wedge \varphi_p\left(\overrightarrow{\alpha}, \overrightarrow{\beta_1}, \overrightarrow{\beta_2}, ...\right) \rightarrow$$
$$\rightarrow \exists \overrightarrow{\gamma} (\mu\left(\overrightarrow{\gamma}\right) \wedge \psi_p\left(\overrightarrow{\alpha}, \overrightarrow{\gamma}, \overrightarrow{\beta_1}, \overrightarrow{\beta_2}, ...\right)))$$

It is clear that the last formula is true in S_P. since the continuity of the parallels mentioned precludes the existence of a "gap" in any directions.

5.2 Proof that the axiom of S_P is a theorem of S_R

In order to see what we get by translating the axiom

$$
(AS_P1)\qquad \begin{array}{l} (\exists\vec{\alpha}^n)_{n<\omega}(\exists\vec{\beta}^m)_{m<\omega}(\psi(\vec{\alpha}^1,\vec{\alpha}^2,...)\wedge\psi(\vec{\beta}^1,\vec{\beta}^2,...)\wedge \\ \wedge\alpha^i_j=\beta^l_k\wedge\neg\alpha^i_j=\beta^p_q\wedge\forall\gamma(\bigvee_{m,n<\omega}\gamma=\alpha^m_n)), \\ \text{ for } l\neq p, k\neq q \text{ and } i,j,l,k,p,q<\omega \end{array}
$$

into S_R, let us first see how the following formula

$$
\begin{array}{l} \psi(\vec{\alpha}^1,\vec{\alpha}^2,...)\Leftrightarrow_{def}\bigwedge_{j\leq 10}\bigwedge_{i<\omega}\varphi_j(\vec{\alpha}^i)\wedge\neg\exists x,y(\alpha^i_n=x=y=\alpha^j_m), \\ \text{for } i,j,m,n<\omega \text{ and } i\neq j \end{array}
$$

of the system S_P look like in S_R:

$$
\begin{array}{l} \psi_r(\vec{a}^1,\vec{a}^2,...)\Leftrightarrow_{def} a^1_1\lessdot^* a^1_2\wedge a^1_2\lessdot^* a^1_3\wedge ...\wedge \\ \wedge a^2_1\lessdot^* a^2_2\wedge a^2_2\lessdot^* a^2_3\wedge ...\wedge\neg\exists x,y(a^i_n=x=y=a^j_m), \\ \text{for } i,j,m,n<\omega \text{ and } i\neq j \end{array}
$$

To see what this means, let us first remember that the supposed function f_2 tied to the translation rule T_2 implies that we always speak about a particular line selected by this function so that it holds that according to T_2 the translation rule $\chi(\alpha_1,\alpha_2,...)=^{T_2} a_1\lessdot^* a_2\wedge a_2\lessdot^* a_3\wedge ...$ says that in S_R we speak only of the regions represented by *diagram 21*, that is, of the regions constituting that which is a line in S_R. The formula $\psi_r\left(\vec{a}^1,\vec{a}^2,...\right)$ speaks of lines being "parallel" by asserting that no two regions from the defining set of regions making up a line can be identical, for otherwise the lines defined by them would share a common part, however small.

In view of all this, the axiom (AS_P1) reads in S_R as follows:

$$
\begin{array}{l} (\exists\vec{a}^n)_{n<\omega}(\exists\vec{b}^m)_{m<\omega}(\psi_r(\vec{a}^1,\vec{a}^2,...)\wedge\psi_r(\vec{b}^1,\vec{b}^2,...)\wedge \\ \wedge a^i_j\mid b^l_k\wedge\neg a^i_j\mid b^p_q\wedge\forall c(\bigvee_{m,n<\omega} c\mid a^m_n)), \\ \text{for } l\neq p, k\neq q \text{ and } i,j,l,k,p,q<\omega \end{array}
$$

The crucial but also a rather tricky thing is to see exactly why all the axioms of S_R are needed to secure the truth of the translation of (AS_P1). Quite generally, the connectedness axioms (AS_R1) - (AS_R3) are needed because they express primitive properties of regions without which it would obviously be impossible to speak of points and lines. In particular, the axiom (AS_R4) precludes the possibility that two disconnected "parts" of one region make up one and the same line; the axioms (AS_R5) - (AS_R6) guarantee the existence of infinitely many regions needed for the very formulation of the axiom (AS_P1); the axiom (AS_R7), which precludes the existence of doughnut-like regions, is needed to secure the connectedness of every region to some region of which the translation of (AS_P1) speaks; the axiom (AS_R8) is necessary for precluding the possibility that the lines the axiom (AS_P1) is speaking about are skew; and finally, the axiom (AS_R9) secures the continuity of regions necessary for the continuity of lines.

6 Metalogical and Metaontological Consequences

In view of the above results, two general conclusions, one metalogical and one metaontological, can be straightforwardly derived.

From the metalogical point of view, it follows that the two formal theories, the point-based and the region-based theory, have the same truth-expressive power in relation to the representation of the infinite two-dimensional continuum. So, it hardly makes any sense to raise, at least *ceteris paribus*, the metalogical question of whether it is the point-based rather than the region-based or the region-based rather the point-based theory that which represents *the true theory* of the infinite two-dimensional continuum. Given that all truths and only truths of any of the two theories can be expressed as truths of any of the theories, it follows that if any of the two theories is true, the other one is true as well.

From the metaontological point of view, it follows that the infinite two-dimensional continuum can be analyzed by starting with a set of null-dimensional points, with one-dimensional lines and two-dimensional regions supervening on them, as well as by starting with a set of two-dimensional regions, with one-dimensional lines and null-dimensional points supervening on them. So again, it hardly makes any sense to raise, at least *ceteris paribus*, the metaontological question of whether the points rather than regions or the regions rather than points represent *the basic elements* of the infinite two-dimensional continuum. Given that regions can be said to supervene on points just as it can be said that points supervene on regions, there can be *no ontological priority* in view of these two kinds of entities.

Moreover, given the translation rules (T_2) and(T_2^*), according to which lines are to be spoken of in S_P and S_R respectively, along with the way in which it is shown by Arsenijević and Kapetanović[12] how points can be spoken of in the Interval-Based System S_I and the way in which two sets of parallels are used above for speaking of the regions of the infinite plain, we can even generalize the story and conclude that, metalogically, the Point-Based System S_P, the Interval-Based System S_I and the Region-Based System S_R, as formal theories of the two-dimensional continuum, are all equivalent in view of their truth-expressive power, while, metaontologically, there can be no ontological priority within the infinite two-dimensional continuum between null-dimensional points, one-dimensional intervals and two-dimensional regions.

In view of all this, it is curious why the Aristotelian and the Cantor theory of the continuum (or of the one-dimensional and two-dimensional continua at least) are still being considered as two rival theories, while the struggle between Gunkologists and Pointillists has been always cited as ontologically important disagreement. It is true, from a historical point of view, that original Aristotelian theory turned out to be insufficient in view of the fact that it lacked the second Cantor condition for the one-dimensional continuity[13], which is expressed above through φ_9 and φ_{10} which are defined in section (3) above. But since this con-

[12]See Arsenijević and Kapetanović (2008a) and Arsenijević and Kapetanović (2008b).
[13]See Cantor (1962), pp. 194-195.

dition has become known, it should have been no problem for Neo-Aristotelians to adjust it in order to complete the Aristotelian Interval-Based System of the linear continuum[14] as well as to complete the Aristotelian Region-Based System of the two-dimensional continuum (as it is done above by the introduction of *the Continuity Axiom* (AS_r9)). It must be that some deep-rooted prejudice has prevented philosophers to find out a peace-making strategy that would enable us to overcome the great struggle between two parties. In what follows, we will try to give a diagnosis of the phenomenon.

6.1 Rejection of Quine's Semantico-Ontological Slogan

Due to the fact that Aristotle accepted that which he called *Zeno's Axiom*, which says that no entity of a higher dimension can consist of entities of a lower dimension, he considered the three-dimensional bodies as basic entities, with surfaces, lines and points as limits supervening on them. Analogously, he considered periods as basic elements of time, with instants only supervening on them. One can understand that after Cantor's introduction of the second continuity condition, which makes it possible to say that the entities of higher dimensions consist of null-dimensional entities, both mathematicians and philosophers have become prone to accept his analysis of the continuum as the right one, since null-dimensional entities look as the best candidates to be considered as proper elements of the continuum, given that only they do not contain either further null-dimensional entities or any other entities as their parts.[15] But this cannot be the whole explanation of the fact that no deeper comparison between the two theories of the continuum has been done.

When in the last three decades of the last century a considerable number of

[14] As it is done in Arsenijević and Kapetanović (2008a) and Arsenijević and Kapetanović (2008b).

[15] See, for instance, Russell (1903), Ch. XXXV, Russell (1914), Ch. V, Carnap (1928), 1.4, Grünbaum (1952), Grünbaum (1974), Ch. 6, Salmon (1975), Ch. 1, Robinson (1989), Lewis (1994) and Earman and Roberts (2006).

philosophers tried to revive the Aristotelian approach, both within philosophy of time as well as within philosophy of space, they did it, at least tacitly, as if the Aristotelian theory should be reconsidered as an alternative to the Cantor one, and not in order to investigate the question of their possible equivalence.[16] True, van Benthem proclaimed that "systematic connections between point structures and period structures enable to use both perspectives at will".[17] But he didn't offer any new logico-ontological framework within which these "systematic connections" are to be understood and, in particular, he didn't raise the direct question of whether there is a clearly definable sense in which the two theories could be said to be equivalent.

There are two related reasons why the question of the possible equivalence between the two theories has not been further investigated. The first reason has directly to do with Quine's famous semantico-ontological slogan, which says that "to be assumed as an entity is to reckoned as the value of a variable".[18] The second reason has to do with the fact that the conception of supervenience was, and still has been, applied nearly exclusively within philosophy of mind, when the mental is said to supervene on the physical but not vice versa, so that the cases of the mutual supervenience, where two sets of entities mutually supervene on each other, has never come into the focus of consideration. Let us explain!

If two theories are such that their variables can range in no model over the elements of one and the same basic set, they represent, according to *Quine's Slogan*, they are about two hopelessly different ontologies, because what exists according to one of them does not exist according to the other, and vice versa.

[16]See, for instance, Hamblin (1969), Hamblin (1971), Humberstone (1979), Foldes (1980), Needham (1981), Burgess (1982), Comer (1985), White (1988), Bochman (1990a), Bochman (1990b), van Benthem (1991), Ch. I.3., and van Benthem (1995). Particularly important are the articles of Roeper's (1997) and (2006), given that they concern more-dimensional Aristotelian continua. See also Mormann (2000) and Sambin (2003).

[17]van Benthem (1991), p. 84.

[18]Quine (1961), p.13.

The same consequence is present in Model Theory.[19] Quine's Slogan represent the basis for standard differentiation between formal theories that are only notationally different amongst themselves and those which are not. So, according to standard view, the Aristotelian and the Cantor theory of the continuum represent real alternatives and not just trivially different theories.

As for the reason concerning supervenience, it may seem that the supervenience relation cannot be symmetric. Some set of entities supervenes on some other set of entities just because the latter is the supervenience base on which the former supervenes, and not vice versa. So, even if the theory of supervenience allows us to say that more-dimensional entities also exist somehow in the Cantorian model, they exist in a sense which is different from the sense in which null-dimensional entities exist. If we wanted to accommodate Quine's Slogan to fit in the theory of existence that is involved in the theory of supervenience, we could say that Quine's Slogan concerns that which exists irreducibly, on which everything else that exists, in a secondary sense, is theoretically and ontologically reducible. But even then, the Aristotelian and the Cantorian theory would be, from the very beginning, logically and ontologically non-trivially different, since the supervenience bases of the two theories are radically different.

However, let us consider the following analogy. Suppose that we compare two formalizations of Propositional Calculus, for instance the Hilbert-Ackermann and the Nicod formalization, whose axioms are different. Now, though the two formalizations do not differ in view of all the truths they express, one could suggest that, in fact, they represent two non-equivalent theories, because they differ in view which truths are basic and which are only derived. Of course, we would reject such a suggestion as silly and claim that the difference between basic and derived truths is irrelevant in the given case. Consequently, we would continue to hold that the difference in formalization does not mean that the

[19] see Hodges (1993), pp. 1-2.

Hilbert-Ackermann[20] and the Nicod[21] formalization represent different theories of Propositional Calculus. But why the difference in view of basic truths doesn't matter in the given case, while the difference in view of basic entities should matter in the case of Gunkology and Pointillism, given that the relation between basic and derived truths is asymmetric just as it is the relation between basic and supervening entities?

There is hardly any essential difference between the two cases, or at least we don't see any. The only reason for remaining stubborn and claiming that there is still an essential difference between the two has only to do with the prejudice condensed in Quine's Semantico-Ontological Slogan, and that is the reason why it ought to be rejected. For our purposes, instead of Quine's Slogan, we should rather accept the Slogan cited more times above: *What matters by comparing two possibly equivalent theories is not the isomorphism or non-isomorphism of the basic sets of their models but the equivalence or non-equivalence of their truth-expressive powers.* This is the essence of Arsenijević's generalized definition of the syntactically and semantically trivial difference between formal theories, which was used above in the proof that the Point-Based and the Region-Based Theory of the two-dimensional continuum, and two ontologies, Pointillism and Gunkology, as two respective mutually supervening models of them, are only trivially different amongst themselves.

6.2 Refutation of the Argument of Arntzenius' and Hawthorne's

Generally, the authors who tried to re-establish the Aristotelian theory of the continuum, several of them cited above,[22] assumed from the very beginning

[20] See Hilbert and Ackermann (1968), p. 27.
[21] See Nicod (1917).
[22] See n. 15.

that the Aristotelian and the Cantorian theory are more than trivially different. However, Arntzenius and Hawthorne offered an *argument*[23] that it must be so, which is prima facie convincing and which therefore should be analyzed and refuted.

The argument is based on what Arntzenius and Hawthorne call *No-Zero Assumption*,[24] to which Gunkologists are allegedly committed. Namely, since "a thing is gunky just in case every part of that thing has proper parts", and since points do not have proper parts, it seems that the gunkologist conception of the continuum implies that there can be no difference between open (or half-open) and closed intervals or between open and closed regions. And then, since the pointillists can speak of such a difference, it follows that there is a real and irreducible difference between Gunkology and Pointillism.

The argument is wrong, because though it is true that regions in the Region-Based Theory are *originally* neither open nor closed, we can yet speak (as we did above) of that which is the difference between open and closed regions according to the Point-Based Theory. For instance, an infinite set of regions of S_R that exhausts completely a given region (cf., for instance, the way in which the axiom (AS_R9) is introduced above) is exactly that which is an open region in S_P, while, *in contrast* to such an infinite set of regions, the single region itself that is exhausted by the given set of regions, though originally neither open nor closed, turns out to be exactly that which is a closed region in S_P.

It is important to note that the difference between closed and open regions in S_R depends on the presence of *supervening* entities in its model. But *the same* is true of S_P, because the regions *supervene* on points as on basic elements of its model. So, the difference between closed and open regions is by no means more basic in S_P than it is in S_R.

[23]See Artzenius and Hawthorne (2005).
[24]Ibid, p. 443.

Probably, the strongest intuitive appeal of Arntzenius' and Hawthorne's *No-Zero Assumption* consists in the fact that in S_P we can get a topologically non-homeomorphic figure by dropping out one single point from a given figure. For instance, if we drop out one single point from a set of points that makes up a circle, we get an open line, which is topologically non-homeomorphic with the given circle. This seems impossible to do in S_R. But, unexpectedly, it is exactly that which we managed to do above by using the "tricky device", when we were defining a circle within S_R via the set of the lines such that just one point of each of them is dropped out (see *diagram 23* and further on).

The wrongness of the Argument of Arntzenius' and Hawthorne's is based on the false belief that due to the fact that points are not *elements* or *parts* of gunky continua, they are simply non-entities in the gunkologist conception of the continuum. That's why Arntzenius thinks that Aristotle held that "there are no instants in time",[25] as well as no points in space [26] which is straightforwardly wrong. When speaking of *Zeno's Axiom*,[27] Aristotle did accept that entities of a higher dimension do not consist of entities of a lower dimension,[28] but he explicitly rejected that they are *nothing*.[29] It was perhaps Zeno[30] who, by arguing against points as constituents of the magnitude, wanted to pass from "Nothing is added" (because nothing has changed in view of the increasement of a given segment or region) to "That which is added is nothing"[31], but Aristotle rejected this second reading of τὸ προσγινόμενον οὐδέν ἐστιν ("nothing is added")[32] and accepted that instants, points, lines and surfaces, though not entities in the primary sense (πρῶτον) still exist in a secondary sense.[33] They exist as limits,

[25]See Arntzenius (2000), p. 187. and p. 202.
[26]See See Arntzenius (2012), Ch. 4.
[27]See Aristotle (1831), *Met.* 1001 b 7.
[28]Ibid loc. cit, 1001 b 7.
[29]Ibid loc. cit, 1001 b 7 ff.)
[30]*DK* B **2.**
[31]See Fränkel (1942), p. 199 ff.
[32]*DK* B **2** 13.
[33]For more about this, see Arsenijević, Šćepanović and Massey (2008), pp. 23 ff.

or, in modern terminology, they supervene on the entities whose limits they are. But independently of Zeno and Aristotle, if we got rid of entities supervening on regions in the models of S_R, we could not speak of lines and regions in the models of S_P as well, because in S_P we have only *points* as the elements of the basic set and not *sets* of points. The power set of the elements of the domain of S_P as well as its elements are not the elements of the domain itself. That's why, in order to see clearly that there is a complete symmetry between S_P and S_R in view of supervening entities, it was so important to formulate the axioms of S_P and S_R in a pure Hilbertian manner, by letting variables range over individuals only, as we have done above.

So, all in all, we should accept that Gunkology and Pointillism are mutually supervening models of the Region-Based and the Point-Based Theory as two only trivially different theories of the two-dimensional continuum.

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