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Generalized concepts of syntactically and semantically trivial differences and instant-based and period-based time ontologies

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Abstract

It is argued that there are interesting cases in which two axiomatic systems syntactically and semantically non-trivially different in the standard sense are rather to be classified as only trivially different. For a strict comparison of the systems of such a kind the generalized concepts of syntactically and semantically trivial differences are formally defined. As an example, it is shown that an instant-based time system and a period-based time system sketched in the paper are just trivially different in the defined sense in spite of the fact that, contrary to Quine's famous requirement, the variables of the two systems can never range over the elements of one and the same basic set. What is the same in relation to both systems is time topology.
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By comparing first-order axiomatic systems according to Quine's semantical formula 'To be assumed as an entity [...] is to be reckoned as the value of a variable' [17, p. 13], one should say that two axiomatic systems not interpretable in such a way that variables of one of them and the variables of the other one range over the elements of one and the same basic sets are both syntactically and semantically non-trivially different.

However, there are interesting cases in which two systems different in the cited sense are more naturally to be taken as only trivially different. That's why more generalized concepts of syntactical and semantical differences are needed. It will be shown of what importance they can be for comparing rival ontologies.

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1. Let us consider two simple first-order systems S_1 and S_2. S_1 contains x_1 and x_2 as the only two individual variable symbols and $\neq$ as the only relation symbol. S_2 contains y_1 and y_2 as the only two individual variable symbols and D as the only relation symbol. Let the elementary *wffs* of S_1 be $x_1 \neq x_1$, $x_1 \neq x_2$, $x_2 \neq x_1$, $x_2 \neq x_2$ and the elementary *wffs* of S_2 be Dy_1y_1, Dy_1y_2, Dy_2y_1, Dy_2y_2, the complex *wffs* being introducable in an obvious way by adding logical constants and quantifiers. Let the axiom scheme of S_1 be $(x_n)\neg x_n \neq x_n$ $(n = 1, 2)$ and the axiom scheme of S_2 be $(y_n)\neg Dy_ny_n$ $(n = 1, 2)$ (in addition to the axioms of propositional and predicate calculi). S_1 and S_2 are *syntactically* only trivially different because there is a set of syntactic constraints, C, which provides a translation of each *wff* of S_1 into just one *wff* of S_2, and *vice versa*, such that each theorem of S_1 is translated into just one theorem of S_2, and *vice versa*. (If the two languages in which S_1 and S_2 are formulated are not to be mixed, all *wffs* of one of the two systems should be replaced *at the same time* through the corresponding *wffs* of the other one.)

In view of the intended generalization, it should be noted that the mutual interchangeability between all the *wffs* of S_1 and all the *wffs* of S_2 is based on a 1–1 mapping of each variable symbol of one of the two systems to a variable symbol of the other one, say of x_1 to y_1 and x_2 to y_2, so that the base for the recursive definition of all syntactic constraints providing interchangeability can be introduced as $x_m \neq x_n =^C Dy_my_n$ $(m = 1, 2;$ $n = 1, 2)$, $=^C$ meaning that the formulae flanking it on each side are interchangeable under C-constraints (translation rules).

From a *semantical* point of view, S_1 and S_2 are equivalent *in relation to* the two respective classes of their models, because each relational structure that is a model for S_1 is a model for S_2, and *vice versa*.

Now, the interesting cases we are looking for are those cases in which there is no such thing possible as a complete mutual translatability between the *wffs* of two (complete first-order) systems S and S' based on one set of syntactic constraints alone, but where there are still *two* recursively definable sets of mutually *non-inverse* translation rules C and C^* (based on mappings of each variable of S to just one variable *tuple* of S' and of each variable of S' to just one variable *tuple* of S), such that all the *wffs* of S as well as all the *wffs* of S' are translatable according to C and C^* respectively, the theorems being always translated into theorems and non-theorems into non-theorems. A concrete example will be formally elaborated in Section 2. In this section shall cite the necessary and sufficient conditions that are to be met in relation to any two complete first-order axiomatic systems containing denumerable infinite sets of variables if the systems are to be understood as syntactically and semantically trivially different in a non-standard yet *generalized sense*, which doesn't mean that the same idea is not naturally extendable to other kinds of systems too.

Supposition 1. Let **L** be a family of complete first-order languages, each of the languages containing a set of individual variable symbols and predicate/relation symbols in addition to logical constants and quantifiers.

Let us pick out from **L** a pair of languages, L and L', whose respective sets of variable symbols are $V = \{v_1, v_2, \ldots, v_n, \ldots\}$ and $V' = \{v'_1, v'_2, \ldots, v'_n, \ldots\}$, and whose respective sets of *wffs* are $\mathcal{F} = \{F_1, F_2, \ldots, F_n, \ldots\}$ and $\mathcal{F}' = \{F_1', F_2', \ldots, F_n', \ldots\}$, where for V and V' two variable symbol mappings f and f^* and for $\mathcal{F}$ and $\mathcal{F}'$ two related sets of syntactic constraints C and C^* are definable as follows:

f is a 1–1 mapping of each variable symbol of L to a $(k+1)$-tuple of variable symbols of L', whereas f^* is a 1–1 mapping of each variable symbol of L' to a $(l+1)$-tuple of variable symbols of L ($k > 0$, $l > 0$), namely:

$$\begin{aligned} f: v_n \to \langle v'_m, v'_{m+1}, \ldots, v'_{m+k}\rangle, \quad & \text{where for } n = 1, 2, 3, \ldots, i, \ldots,\ m = 1, k+2, \\ & 2k+3, \ldots, (i-1)k+1, \ldots \text{ resp. } (k > 0), \\ f^*: v'_n \to \langle v_m, v_{m+1}, \ldots, v_{m+l}\rangle, \quad & \text{where for } n = 1, 2, 3, \ldots, i, \ldots,\ m = 1, l+2, \\ & 2l+3, \ldots, (i-1)l+1, \ldots \text{ resp. } (l > 0); \end{aligned}$$

C is a set of syntactic constraints, related to f, defining a 1–1 translation of each *wff* from $\mathcal{F}$ into a *wff* from $\mathcal{F}'$ in a standard recursive way so that for each i and $F_i \in \mathcal{F}$:

$$\begin{aligned} & F_i(v_n, v_{n+p_1}, v_{n+p_2}, \ldots, v_{n+p_q}) \\ & \quad =^C F'_j\big(f(v_n), f(v_{n+p_1}), f(v_{n+p_2}), \ldots, f(v_{n+p_q})\big) \end{aligned}$$

for some $F'_j \in \mathcal{F}'$, where

- $F_i(v_n, v_{n+p_1}, v_{n+p_2}, \ldots, v_{n+p_q})$ denotes F_i and indicates in numerical order (in the parentheses) all and only variable symbols which occur in F_i ($p_1, p_2, \ldots, p_q$ being integers such that $-1 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_q$);
- $F'_j(f(v_n), f(v_{n+p_1}), f(v_{n+p_2}), \ldots, f(v_{n+p_q}))$ denotes some F'_j from $\mathcal{F}'$ which contains all and only variable symbols from the tuples to which the variable symbols from F_i are mapped according to f;
- $=^C$ connects F_i and F'_j by stating that F'_j is to be taken as just that *wff* which is obtained from F_i according to C.
- The set of syntactic constraints C^*, providing a translation of each *wff* from $\mathcal{F}'$ into just one *wff* from $\mathcal{F}$, is to be understood analogously to C.

It is important to notice that, since f and f^* are mappings of variable symbols to $(k+1)$-tuples and $(l+1)$-tuples, where $k \geqslant 1$ and $l \geqslant 1$, the translations under C and C^* are translations of all the *wffs* from $\mathcal{F}$ *into* (not *onto*) $\mathcal{F}'$ and of all the *wffs* from $\mathcal{F}'$ *into* (not *onto*) $\mathcal{F}$ respectively, C and C^* being necessarily *non*-inverse.

Supposition 2. Let two axiomatic systems S and S' be built up by introducing two sets of axioms $\mathcal{A}$ and $\mathcal{A}'$ into L and L' respectively, both $\mathcal{A}$ and $\mathcal{A}'$ containing as their proper subsets all axioms of the propositional and predicate calculi. (The derivation rules are supposed to be standard derivation rules of the propositional and quantification calculi.) Let it be supposed that under f, f^*, C, C^* from Supposition 1 each theorem of S is translated into a theorem of S' as well as each theorem of S' into a theorem of S and that no non-theorem of S is translated into a theorem of S' as well as no non-theorem of S' into a theorem of S.

If Suppositions 1 and 2 are satisfied, a curious situation arises. Although not all of the *wffs* from $\mathcal{F}'$ are obtainable from $\mathcal{F}$ and not all of the *wffs* from $\mathcal{F}$ are obtainable from $\mathcal{F}'$ under the two sets of translation rules, $\mathcal{F}$ contains as its proper subset all the *wffs* from $\mathcal{F}'$ translated into $\mathcal{F}$, and $\mathcal{F}'$ contains as its proper subset all the *wffs* from $\mathcal{F}$ translated into $\mathcal{F}'$, the theorems always being translated into theorems and the non-theorems into non-theorems. But, however curious this can be, it is the latter fact, and not the former, that seems decisive for classifying S and S' as syntactically only trivially different systems (in the generalized sense), vindicating the following definition:

Definition 1. Any two complete axiomatic systems S and S', expressed in two different languages L and L' from the family of first-order languages $\mathbf{L}$, where L and L' are not trivially different languages in the standard sense, are syntactically trivially different in *the generalized sense* if both Suppositions 1 and 2 are satisfied in relation to L, L', S and S'.

Turning to the question of interpretation of systems S and S' syntactically trivially different in the generalized sense, one should notice that (1) although variables of one of them can range over the elements of the basic set in no model of the other one and although syntactic constraints C and C^* are not inverse, (2) S *is* interpretable under C in *any* relational structure (if there is any) *in which* S' is satisfied as well as S' *is* interpretable under C^* in *any* relational structure (if there is any) *in which* S is satisfied. Now, is it the *former* fact (due to which the Quinean slogan is not satisfied) or is it the *latter* one that is to be taken as decisive for the question of whether two systems syntactically trivially different in the generalized sense (according to Definition 1) are also to be taken as semantically only trivially different (in the generalized sense) in relation to the respective classes of their models? The answer to this question seems to require some kind of informal philosophical argument, which is going to be dealt with in Section 3. At this stage we shall simply classify such systems as semantically trivially different in the generalized sense in relation to the respective classes of their models, where the qualification 'in the generalized sense' is to be understood *technically*, as meaning just that each of the two systems syntactically trivially different according to Definition 1 is always interpretable in a corresponding model of the other one *after* an appropriate translation. Therefore:

Definition 2. Two systems having models and being syntactically trivially different according to Definition 1 are semantically trivially different in *the generalized sense* in relation to the two respective classes of their models.

It is easy to see how the *standard* conception of syntactically and semantically trivial differences related to the first-order systems can be obtained as a *special* case when only Supposition 1 is slightly changed as to be satisfied under syntactic constraints which are *inverse*, the translations of all the *wffs* of L into *wffs* of L' and of all the *wffs* of L' into *wffs* of L being then 1–1 mappings of $\mathcal{F}$ *onto* $\mathcal{F}'$ and of $\mathcal{F}'$ *onto* $\mathcal{F}$ respectively. Actually, k and l from the above definitions of f and f^* should be taken to be equal to zero.

2. The importance of the above generalizations will be now illustrated with an intriguing example concerning 'the great struggle' about the structure of time,[1] one of the two parties holding *instants*, the other one *periods* as its basic stuffs.[2]

For the sake of simplicity I shall use the axiomatization whose models are perfect but not coherent sets in Cantor's sense[3].

Let S_P contains—besides logical constants $\neg$, $\Rightarrow$, $\wedge$, $\vee$ and $\Leftrightarrow$—individual variable symbols $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ $(n = 1, 2, \ldots)$ quantifiable by universal and existential quantifiers. The variables are supposed to range over durationless pointlike instants. Also let S_P contain relation symbols $\equiv$ and $<$, to be interpreted as identity and precedence relation respectively[4]. Let elementary *wffs* be $\alpha_m \equiv \alpha_n$ and $\alpha_m < \alpha_n$ $(m = 1, 2, \ldots; n = 1, 2, \ldots)$ and the axiom schemes of S_P be (in addition to the axioms of propositional and quantificational logic):

(AP1) $(\alpha_n)\neg\alpha_n < \alpha_n$;
(AP2) $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l < \alpha_m \wedge \alpha_m < \alpha_n \Rightarrow \alpha_l < \alpha_n)$;
(AP3) $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \vee \alpha_n < \alpha_m \vee \alpha_m \equiv \alpha_n)$;
(AP4) $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_l < \alpha_n \Rightarrow \alpha_m < \alpha_n)$;
(AP5) $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_n < \alpha_l \Rightarrow \alpha_n < \alpha_m)$;
(AP6) $(\alpha_m)(\exists\alpha_n)\alpha_m < \alpha_n$;
(AP7) $(\alpha_m)(\exists\alpha_n)\alpha_n < \alpha_m$;
(AP8) $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \Rightarrow (\exists\alpha_l)(\alpha_m < \alpha_l \wedge \alpha_l < \alpha_n))$;
$(l = 1, 2, \ldots; m = 1, 2, \ldots; n = 1, 2, \ldots)$.

Let S_I contains—besides logical constants $\neg$, $\Rightarrow$, $\wedge$, $\vee$ and $\Leftrightarrow$—individual variable symbols $a_1, a_2, \ldots, a_n, \ldots$ $(n = 1, 2, \ldots)$ quantifiable by universal and existential quantifiers. The variables are supposed to range over time-intervals with duration (periods). Also let S_P contain relation symbols $=$, $\prec$, $\lessdot$, $\cap$ and $\Subset$, to be interpreted as identity, precedence, abutment, overlapping and inclusion relation respectively. Let elementary *wffs* be $a_m = a_n$,

[1] A 'great struggle' ('ein großer Streit') is Cantor's expression that he used for characterizing the rivalry between the two approaches related to the question about continua in general (cf. [10, p. 190]). Van Benthem suggests, however, that 'it would be a philosophical perversion to [re-]open hostilities' [4, p. 8]. A more ambitious task, we are now dealing with, is to show that the differences are syntactically and semantically trivial and, therefore, ontologically inessential.

[2] Due to the general acceptance of what Aristotle has denoted as 'Zeno's axiom', according to which entities of a higher dimension cannot be built up out of entities of a lower dimension (cf. [1, 1001 b 7]), the period-based approach dominated until the second half of 19th century. After Dedekind and Cantor, the situation has changed radically, so that now the period-based approach has still been considered revisionist (see [4, p. 2]), in spite of a number of articles favouring this approach (cf. [7–9,13–16,18–20]). In view of what follows, the 'great struggle' should be considered trivialized (at least as concerning the structure of pure time—see Section 3) in relation to any given time topology whatsoever.

[3] I take 'coherent' to be the best translation of 'zusammenhängand' in the context of Cantor's definition of continuum (see [10, p. 194]).

[4] In view of the aim to question the mentioned implication of Quine's semantical formula, the two systems that are to be compared are not sketched as propositional tense logic where the instants and periods are confined to metalanguages (as in [15,18–20]), but in such a way that variables are directly interpretable as ranging over the sets of two related sorts of entities (as in [7–9,13,14,16]).

$a_m \prec a_n$, $a_m \lessdot a_n$, $a_m \cap a_n$, $a_m \Subset a_n$ ($m = 1, 2 \ldots$; $n = 1, 2, \ldots$). In view of what follows, it is, however, to be noted that $\lessdot$, $\cap$ and $\Subset$, although introduced as primitives for the sake of a conventional characterization of period-based time structures[5] as well as for the sake of generality, are definable (at least in view of the system we are here interested in) *via* $=$ and $\prec$ in the following way:

$$a_m \lessdot a_n \underset{\text{def.}}{\Leftrightarrow} a_m \prec a_n \wedge \neg(\exists a_l)(a_m \prec a_l \wedge a_l \prec a_n),$$

$$a_m \cap a_n \underset{\text{def.}}{\Leftrightarrow} (\exists a_l)(\exists a_k)(a_l \prec a_n \wedge \neg a_l \prec a_m \wedge a_m \prec a_k \wedge \neg a_n \prec a_k),$$

$$a_m \Subset a_n \underset{\text{def.}}{\Leftrightarrow} \neg a_m = a_n \wedge (a_l)(a_l \cap a_m \Rightarrow a_l \cap a_n)$$

$$(k = 1, 2, \ldots;\ l = 1, 2, \ldots;\ m = 1, 2, \ldots;\ n = 1, 2, \ldots).$$

Now, let in addition to the axioms of propositional and quantificational logic, the axiom schemes of S_I be:

($\mathrm{A_I}$1) $(a_n)\neg a_n \prec a_n$;
($\mathrm{A_I}$2) $(a_k)(a_l)(a_m)(a_n)(a_k \prec a_m \wedge a_l \prec a_n \Rightarrow a_k \prec a_n \vee a_l \prec a_m)$;
($\mathrm{A_I}$3) $(a_m)(a_n)(a_m \prec a_n \Rightarrow a_m \lessdot a_n \vee (\exists a_l)(a_m \lessdot a_l \wedge a_l \lessdot a_n))$;
($\mathrm{A_I}$4) $(a_k)(a_l)(a_m)(a_n)(a_k \lessdot a_m \wedge a_k \lessdot a_n \wedge a_l \lessdot a_m \Rightarrow a_l \lessdot a_n)$;
($\mathrm{A_I}$5) $(a_k)(a_l)(a_m)(a_n)(a_k \lessdot a_l \wedge a_l \lessdot a_n \wedge a_k \lessdot a_m \wedge a_m \lessdot a_n \Rightarrow a_l = a_m)$;
($\mathrm{A_I}$6) $(a_m)(\exists a_n)a_m \prec a_n$;
($\mathrm{A_I}$7) $(a_m)(\exists a_n)a_n \prec a_m$;
($\mathrm{A_I}$8) $(a_m)(\exists a_n)a_n \Subset a_m$;
$(k = 1, 2, \ldots;\ l = 1, 2, \ldots;\ m = 1, 2, \ldots;\ n = 1, 2, \ldots)$.

Lemma 1. *If* $\langle \Re_P, \equiv^*, <^* \rangle \vDash S_P$—$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ *ranging over the elements of* $\Re_P$, *and* $\equiv$ *and* $<$ *being interpreted as* $\equiv^*$ *and* $<^*$ *respectively—then*

$$\langle \Re', =^*, \prec^*, \lessdot^*, \cap^*, \Subset^* \rangle \vDash S_I,$$

given that

$$\Re' = \{\langle x_i, x_j \rangle \mid x_i \in \Re_P \;\&\; x_j \in \Re_P \;\&\; x_i <^* x_j\}$$

and

$$\langle x_i, x_j \rangle =^* \langle x_k, x_l \rangle \underset{\text{def.}}{\Leftrightarrow} x_i \equiv^* x_k \wedge x_j \equiv^* x_l,$$

$$\langle x_i, x_j \rangle \prec^* \langle x_k, x_l \rangle \underset{\text{def.}}{\Leftrightarrow} x_j \equiv^* x_k \vee x_j <^* x_k,$$

$$\langle x_i, x_j \rangle \lessdot^* \langle x_k, x_l \rangle \underset{\text{def.}}{\Leftrightarrow} x_j \equiv^* x_k,$$

$$\langle x_i, x_j \rangle \cap^* \langle x_k, x_l \rangle \underset{\text{def.}}{\Leftrightarrow} x_i <^* x_k \wedge x_k <^* x_j \wedge x_j < x_l,$$

$$\langle x_i, x_j \rangle \Subset^* \langle x_k, x_l \rangle \underset{\text{def.}}{\Leftrightarrow} (x_k <^* x_i \wedge \neg x_l <^* x_j) \vee (x_j <^* x_l \wedge \neg x_i <^* x_k),$$

[5] Cf. [6, I. 3].

$a_1, a_2, \ldots, a_n, \ldots$ ranging over the elements of $\Re'$, and $=$, $\prec$, $\lessdot$, $\cap$, $\Subset$ being interpreted as $=^*$, $\prec^*$, $\lessdot^*$, $\cap^*$, $\Subset^*$ respectively.[6]

Proof. Let us check (A_I1). $(a_n)\neg a_n \prec a_n$ would not be satisfied in $\langle \Re', =^*, \prec^*, \lessdot^*, \cap^*, \Subset^* \rangle$ only if there were x_i, x_j, $x_i \prec^* x_j$, so that $\langle x_i, x_j \rangle \prec^* \langle x_i, x_j \rangle$. However, this means, according to the above definitional equivalence concerning $\langle x_i, x_j \rangle \prec^* \langle x_k, x_l \rangle$, that it should be $x_i \prec^* x_j \wedge (x_j \equiv^* x_i \vee x_j <^* x_i)$, which is impossible, because

$$\vdash \neg(\exists a_m)(\exists a_n)\big(a_m < a_n \wedge (a_n \equiv a_m \vee a_n < a_m)\big),$$

and $\langle \Re_P, \equiv^*, <^* \rangle$ is supposed to be a model for S_P.

It can be shown in a similar way that each of the axioms of S_I is satisfied in $\langle \Re', =^*, \prec^*, \lessdot^*, \cap^*, \Subset^* \rangle$, where for the axioms ($A_I2$), ($A_I3$), ($A_I4$), ($A_I5$) and ($A_I8$) the cited definitions of $\lessdot$, $\cap$ and $\Subset$ are to be used in addition to the definitional equivalences concerning $=^*$, $\prec^*$, $\lessdot^*$, $\cap^*$, $\Subset^*$.[7] □

Lemma 2. *If $\langle \Re_I, =^*, \prec^*, \lessdot^*, \cap^*, \Subset^* \rangle \vDash S_I$—$a_1, a_2, \ldots, a_n, \ldots$ ranging over the elements of $\Re_I$, and $=$, $\prec$, $\lessdot$, $\cap$, $\Subset$ being interpreted as $=^*$, $\prec^*$, $\lessdot^*$, $\cap^*$, $\Subset^*$ respectively—then*

$$\langle \Re'', \equiv^*, <^* \rangle \vDash S_P,$$

given that

$$\Re'' = \big\{\langle \mathcal{M}_i, \mathcal{M}_j \rangle \mid \mathcal{M}_i \subset \Re_I \;\&\; \mathcal{M}_j \subset \Re_I \;\&\; (y_i)(y_j) \\ (y_i \in \mathcal{M}_i \;\&\; y_j \in \mathcal{M}_j \Rightarrow y_i \prec^* y_j)\big\}$$

and

$$\langle \mathcal{M}_i, \mathcal{M}_j \rangle \equiv^* \langle \mathcal{M}_k, \mathcal{M}_l \rangle \underset{\text{def.}}{\Leftrightarrow} \mathcal{M}_i = \mathcal{M}_k,$$

$$\langle \mathcal{M}_i, \mathcal{M}_j \rangle <^* \langle \mathcal{M}_k, \mathcal{M}_l \rangle \\ \underset{\text{def.}}{\Leftrightarrow} (y_i)(y_l)(y_i \in \mathcal{M}_i \;\&\; y_l \in \mathcal{M}_l \Rightarrow y_i \prec y_l \;\&\; \neg y_i \lessdot y_l),$$

$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ ranging over the elements of $\Re''$, and $\equiv$ and $<$ being interpreted as $\equiv^*$ and $<^*$ respectively.

Proof. The proof is *mutatis mutandis* the same as the proof of Lemma 1. □

Metatheorem 1. *S_P and S_I are syntactically trivially different in the generalized sense.*

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[6] Due to the fact that the elements of the basic set of a model for S_I are 'stretches' that can be treated as 'intervals between distinct points' only if they are *not* conceived as *sets* of points but as something unequivocally defined by pairs of distinct points, it would be misleading *in the given context* to conceive periods (as in [11,12]) as closed intervals. Originally, periods should not be said to be either open or closed. However, the standard difference between open and closed intervals can be spoken of in a period-based system too, by defining open intervals as infinite sequences of abutting periods having a *supremum* or an *infinum*. Only then, in contrast to such infinite *sequences* of periods, *periods* can be said to be closed intervals (cf. [2, p. 171]).

[7] The two related systems being complete, the checking of the axioms suffices.

Proof. Let

$$f : \alpha_n \to \langle a_{2n-1}, a_{2n} \rangle \quad (n = 1, 2, \ldots),$$
$$f^* : a_n \to \langle \alpha_{2n-1}, \alpha_{2n} \rangle \quad (n = 1, 2, \ldots).$$

Let syntactic constraints C_1–C_5 and C_1^*–C_5^*, providing a 1–1 translation of the set of all the *wffs* of S_P into a subset of *wffs* of S_I and the set of all the *wffs* of S_I into a subset of *wffs* of S_P respectively, be defined as follows:

C_1 $\alpha_n \equiv \alpha_m =^C a_{2n-1} \lessdot a_{2n} \wedge a_{2m-1} \lessdot a_{2m} \wedge a_{2n-1} \lessdot a_{2m}$,

C_2 $\alpha_n < \alpha_m =^C a_{2n-1} \lessdot a_{2n} \wedge a_{2m-1} \lessdot a_{2m} \wedge a_{2n-1} \prec a_{2m} \wedge \neg a_{2n-1} \lessdot a_{2m}$,

C_3 $\neg F_P =^C \neg C(F_P)$, where F_P is a *wff* of S_P translated according to C_1–C_5 into *wff* $C(F_P)$ of S_I,

C_4 $F'_P \heartsuit F''_P =^C C(F'_P) \heartsuit C(F''_P)$, where $\heartsuit$ stands for $\Rightarrow$ or $\wedge$ or $\vee$ or $\Leftrightarrow$, and F'_P and F''_P stand for two *wffs* of S_P translated according to C_1–C_5 into two *wffs* of S_I, $C(F'_P)$ and $C(F''_P)$ respectively,

C_5
$$\mathbf{F_P}\big(\varphi(Q_1\alpha_n)\chi(Q_2\alpha_m)\psi F_{P_S}(\alpha_n, \alpha_m)\omega\big)$$
$$=^C \mathbf{C(F_P)}\big(C(\varphi)(Q_1 a_{2n-1})(Q_1 a_{2n})C(\chi)(Q_2 a_{2m-1})(Q_2 a_{2m})C(\psi)$$
$$C(F_{P_S})(a_{2n-1}, a_{2n}, a_{2m-1}, a_{2m})C(\omega)\big),$$

where $\mathbf{F_P}$ is a *wff* of S_P and $\mathbf{C(F_P)}$ is the *wff* of S_I into which $\mathbf{F_P}$ is translated according to C_1–C_5 when the structures of $\mathbf{F_P}$ and $\mathbf{C(F_P)}$, indicated in the parentheses, are such that:

- Q_1 and Q_2 stand for quantifiers,
- F_{P_S} stands for a *wff* of S_P containing α_n and α_m and being translated according to C_1–C_5 into *wff* $C(F_{P_S})$ of S_I containing a_{2n-1}, a_n, a_{2m-1} and a_{2m},
- φ, χ, ψ stand either for nothing or for quantified variables and/or *wffs* of S_P translated according to C_1–C_5 into $C(\varphi)$, $C(\chi)$ and $C(\psi)$ respectively,
- ω stands either for nothing or for a *wff* of S_P translated according to C_1–C_5 into $C(\omega)$;

C_1^* $a_n = a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \alpha_{2n-1} \equiv \alpha_{2m-1} \wedge \alpha_{2n} \equiv \alpha_{2m}$,

C_2^* $a_n \prec a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \neg\alpha_{2m-1} < \alpha_{2n}$,

C_3^* $\neg F_I =^{C^*} \neg C^*(F_I)$, where F_I is a *wff* of S_I translated according to C_1^*–C_5^* into *wff* $C(F_I)$ of S_P,

C_4^* $F'_I \heartsuit F''_I =^{C^*} C^*(F'_I) \heartsuit C^*(F''_I)$, where $\heartsuit$ stands for $\Rightarrow$ or $\wedge$ or $\vee$ or $\Leftrightarrow$, and F'_I and F''_I stand for two *wffs* of S_I translated according to C_1^*–C_5^* into two *wffs* of S_P, $C^*(F'_I)$ and $C^*(F''_I)$ respectively,

C_5^*
$$\mathbf{F_I}\big(R(Q_1 a_n)T(Q_2 a_m)U F_{I_S}(a_n, a_m)W\big)$$
$$=^{C^*} \mathbf{C}^*(\mathbf{F_I})\big(C^*(R)(Q_1\alpha_{2n-1})(Q_1\alpha_{2n})C^*(T)(Q_2\alpha_{2m-1})(Q_2\alpha_{2m})C^*(U)$$
$$C^*(F_{I_S})(\alpha_{2n-1}, \alpha_{2n}, \alpha_{2m-1}, \alpha_{2m})C^*(W)\big),$$

where $\mathbf{F_I}$ is a *wff* of S_I and $\mathbf{C}^*(\mathbf{F_I})$ is the *wff* of S_P into which $\mathbf{F_I}$ is translated according to C_1^*–C_5^* when the structures of $\mathbf{F_I}$ and $\mathbf{C}^*(\mathbf{F_I})$, indicated in the parentheses, are such that:

- Q_1 and Q_2 stand for quantifiers,
- F_{I_S} stands for a *wff* of S_I containing a_n and a_m and being translated according to C_1^*–C_5^* into *wff* $C^*(F_{I_S})$ of S_P containing α_{2n-1}, α_n, α_{2m-1} and α_{2m},
- R, T, U stand either for nothing or for quantified variables and/or *wffs* of S_I translated according to C_1^*–C_5^* into $C^*(R)$, $C^*(T)$ and $C^*(U)$ respectively,
- W stands either for nothing or for a *wff* of S_I translated according to C_1^*–C_5^* into $C^*(W)$.

Now, it can be shown that according to C_1–C_5 each axiom of S_P is translated into a theorem of S_I. This time, let us check (AP2). C(AP2), according to C_2, C_4 and C_5, reads as follows:

$$(a_{2n-1})(a_{2n})(a_{2m-1})(a_{2m})(a_{2k-1})(a_{2k})(a_{2n-1} \lessdot a_{2n} \wedge a_{2m-1} \lessdot a_{2m} \wedge a_{2n-1} \prec a_{2m}$$
$$\wedge \neg a_{2n-1} \lessdot a_{2m} \wedge a_{2m-1} \lessdot a_{2m} \wedge a_{2k-1} \lessdot a_{2k} \wedge a_{2m-1} \prec a_{2k} \wedge \neg a_{2m-1} \lessdot a_{2k}$$
$$\Rightarrow \quad a_{2n-1} \lessdot a_{2n} \wedge a_{2k-1} \lessdot a_{2k} \wedge a_{2n-1} \prec a_{2k} \wedge \neg a_{2n-1} \lessdot a_{2k}).$$

By using Lemma 1, it is easy to see that if C(AP2) were not a theorem, there would be some y_r, y_s, y_t, y_u, y_v, y_w in a model for S_I so that $y_r \lessdot^* y_s$, $y_u \lessdot^* y_t$, $y_v \lessdot^* y_w$, $y_r \prec^* y_t$, *not* $y_r \lessdot^* y_t$, $y_u \prec^* y_w$, *not* $y_u \lessdot^* y_w$. However, this means, by assuming that such a model for S_I is constructed on the base of a model for S_P in the way in which it is done in Lemma 1, that in that model for S_P there are pairs of elements $\langle x_{r_1}, x_{r_2}\rangle$, $\langle x_{s_1}, x_{s_2}\rangle$, $\langle x_{t_1}, x_{t_2}\rangle$, $\langle x_{u_1}, x_{u_2}\rangle$, $\langle x_{v_1}, x_{v_2}\rangle$, $\langle x_{w_1}, x_{w_2}\rangle$, which are ordered by $<^*$ and which correspond to y_r, y_s, y_t, y_u, y_v, y_w respectively, so that $x_{r_2} <^* x_{t_2}$, $x_{t_2} <^* x_{v_2}$, $x_{v_2} \equiv^* x_{w_1}$ and *not* $x_{r_2} <^* x_{w_1}$, which is impossible due to the fact that (AP2) is satisfied in each model for S_P.

After proving, in a similar way, that according to C_1–C_5 each axiom of S_P is translated into a theorem of S_I, it is easy to see that $\vdash C(F_{P_i})$ for each i such that $\vdash F_{P_i}$. Namely, due to C_3 and C_4, the sameness of the general structures of F_{P_i} and $C(F_{P_i})$ guarantees the corresponding applicability of *modus ponens*, whereas, due to C_5, substitutions, generalizations and instantiations related to α_n in a proof for F_{P_i} should be simply followed by the application of the corresponding rules related to a_{2n-1} and a_{2n} in the corresponding proof for $C(F_{P_i})$.

Having proved that according to C_1–C_5 each theorem of S_P is translated into a theorem of S_I, it is still to be proved that no non-theorem of S_P is translated into a theorem of S_I. Now, the two systems being consistent and complete, *not* $\vdash F_{P_i}$ *entails* $\vdash \neg F_{P_i}$ for each i. Therefore, $\vdash C(\neg F_{P_i})$, and so *not* $\vdash C(F_{P_i})$.

What is proved, with the use of Lemma 1, for translations governed by C_1–C_5, can be proved *mutatis mutandis*, by using Lemma 2, for the translations governed by C_1^*–C_5^*, which means that S_P and S_I are syntactically trivially different in the generalized sense. □

Metatheorem 2. *S_P and S_I are trivially different in the generalized sense in relation to the two respective classes of their models.*

Proof. The statement of the metatheorem follows directly from Definition 2 and Metatheorem 1. □

3. The same formal system can be given completely different interpretations (e.g., it can be about lines, atoms, people, etc.). *A fortiori*, two syntactically only trivially different systems need not be semantically trivially different in relation to two respective interpretations. So, even if one is convinced by the reason cited in Section 1 that the two systems S_P and S_I sketched in Section 2 are to be classified as *syntactically* trivially different (in the generalized sense), he can still doubt that S_P and S_I, when interpreted as an *instant* time system and a *period* time system respectively, should be considered as *semantically* trivially different (in the generalized sense, too). In particular, he could argue that the two *intended* models for S_P and S_I are just *as relevantly different as* intended models for any two respective systems of two obviously different interpretations, because instants are *not* periods and periods are *not* instants. So, he could conclude, after all, that though S_P is interpretable in the corresponding period time-structure *after* it has been translated into S_I and S_I is interpretable in the corresponding instant time-structure *after* it has been translated into S_P, we should not classify S_P and S_I as semantically trivially different (in the generalized sense either).

If we want to avoid *begging the question* in this discussion about the *philosophical* plausibility of generalizing the semantically trivial difference, *technically* introduced by Definition 2 in Section 1, we need further, independent arguments.

On the one hand, the instant-based time ontologists can say that the time model for S_P *should* be accepted as the *intended* time model for S_I too, because periods, formally defined in Lemma 1 in Section 2 as ordered pairs of two distinct instants, are *nothing else but* relations between two instants. On the other hand, the period-based time ontologists can say that the time model for S_I *should* be accepted as the *intended* time model S_P too, because instants, formally defined in Lemma 2 in Section 2 as ordered pairs of two equivalence classes of abutting periods, are *nothing else but* relations between abutting periods. The point is that according to the instant-based time ontology we do actually speak about instants and their relations when we speak about periods and their relations while according to the period-based time ontology we do actually speak about periods and their relations when we speak about instants and their relations.

The difference between the 'nothing-else-but'-claims of the instant-based time ontology and the period-based time ontology would we annihilated through the 'as-well-as'-claim stating that we speak *also* about periods and their relations when we speak about instants and their relations *as well as* we speak *also* about instants and their relations when we speak about periods and their relations. According to the 'as-well-as'-claim, the time model for S_P *should* be accepted as the *intended* time model for S_I too, *and also* the time model for S_I should be accepted as the *intended* time model for S_P too.

Now, I want to argue that Definition 2 in Section 1 is also *philosophically* vindicated, because the 'as-well-as'-claim is always plausible *ceteris paribus*. The argument is simple and not circular. If two systems are mutually translatable according to Definition 1, all statements about a model for one of them are expressible as statements about the corresponding model for the other one and *vice versa*, so there can be no *inherent* reason favouring one of the two formulable 'nothing-else-but'-claims over the other. If two related 'nothing-else-but'-claims are formulable in relation to two corresponding relational structures but neither of the claims is favourable, then the corresponding 'as-well-as'-claim is vindicated. If the 'as-well-as'-claim is vindicated, the intended model for one of the two corresponding systems is also the intended model for the other one and *vice versa*. If the intended model for one of the two corresponding systems is the intended model for the other one and *vice versa*, then the two systems are semantically trivially different.

The method used in defining the time model for S_I in terms of a time model for S_P and *vice versa* can be applied to other time systems, independently of whether the time is conceived as discrete, dense or continuous, open or closed, branching or not branching, finite or infinite.[8] It is only the general applicability of this method by constructing a pair of systems trivially different in the precisely defined generalized sense that vindicates van Benthem's proclamation that 'systematic connections between point structures and period structures enable one to use both perspectives at will' [5, p. 84]. So, if the idea of syntactically and semantically trivial differences is applicable in these cases too, it can be said that what is the *same* in relation to any two time axiomatiozations that are trivially different in the generalized sense is time *topology*. If so, it is this *invariance* in view of *topology* that could be said to represent the *positive* account of the trivialization of the semantical differences between the corresponding systems.

Consequently, one might expect that the 'great struggle' between the instant-based and period-based ontologists will turn out to be 'much ado about nothing'. However, it must never be forgotten that the 'as-well-as'-claim is plausible with certainty only under *ceteris paribus* clause. For instance, one can wish to extend two systems, supposedly trivially different in the generalized sense, by introducing predicates denoting some external properties attributable to instants and periods. Now, even if interval predicates in one of the two systems and instants predicates in the other one are definable *via* instant predicates and interval predicates respectively, it doesn't follow that the resulting systems are such that the idea of trivial differences in the generalized sense is extendable to them.[9]

The general lesson is that the answer to the question of whether the difference between two axiomatized rival ontologies, or two formal theories in general, is to be considered inessential or essential does not depend on whether the variables of the two systems can ever range over the elements of one and the same basic set or not, but rather on the applicability of the concept of the syntactically and semantically trivial differences in the generalized sense.

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[8] True, the first and the last instant of a finite time structures cannot be defined *via* two equivalence classes of abutting periods, but they can still be defined *via* equivalence classes of periods which don't abut on any period and on which no period abuts, respectively.

[9] Cf. [3, pp. 188ff].

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