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HOW MANY PHYSICALLY DISTINGUISHED PARTS CAN A LIMITED BODY CONTAIN?

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I THE PROBLEM

Is it possible, in principle at least, for a limited physical body, a cube for instance, to contain, or to consist of, an infinite number of parts of non-infinitesimal magnitude, every two adjacent members of the set of these parts being physically distinguished amongst themselves?

It may seem that the obvious answer to this question is a positive one: a cube can contain, for instance, an infinite number of parallelepipeds arranged in such a manner that the left half of the cube consists of red stuff, the left half of the remainder of the cube of green stuff, the left half of the rest of the cube of red stuff again, and so on according to a geometric progression.

The possible objection of the radical empiricists,[1] who would insist that we should never speak of differently coloured parts of a billionth part of a pin if we want not to violate the limitations imposed by scientific knowledge about the nature of colour, may be avoided if we speak of ‘red’ and ‘green’ parallelepipeds instead of red and green ones, referring by ‘red’ and ‘green’ to some concealed properties of matter, which could possibly be revealed, for which it doesn’t hold that they can’t be properties of a billionth part of a pin, or of any other small portion of matter whatsoever.

Unexpectedly, however, there are great difficulties with the claim that within a given unit cube an infinite number of ‘red’ and ‘green’ parallelepipeds can be arranged simultaneously according to a geometric progression, the lengths of their edges being ½, ¼, 1/8, …, ½”,… respectively (n = l, 2, 3,…).

Let us suppose that there are ‘red’ and ‘green’ properties and that the ‘red’ and ‘green’ portions of matter are really arranged as described. Is the first of the parallelepipeds, observing from right to left, ‘red’ or ‘green’?

The question may be operationalized in many ways. Suppose, for instance, that it is dangerous to look at the ‘red’ stuff and not dangerous to look at the ‘green’, and that any ‘green’ slice, however thin, is a good shield for looking at the ‘red’ stuff. Is it dangerous or not to look at the body consisting only of ‘red’ and ‘green’ parallelepipeds of non-infinitesimal sides arranged according to the cited law?[2]

It may be said that we don’t know what the case may be, but, it seems, it is certain that in any given case this must be either dangerous or not dangerous, because the ‘red’ and ‘green’ parallelepipeds are, by definition, arranged in a non-overlapping manner: when approaching the body from the right, and by just touching it without penetrating it, one would have to be in

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[1] See Hilbert and Bernays [4] p. 16, and Black [1] pp. 116-18.

[2] In this formulation the problem is sufficiently general to be independent of whether it is stated that an infinite number of 'red' and 'green' parallelepipeds of non-infinitesimal magnitude exhausts the whole unit cube or it is stated only that the unit cube contains an infinite number of them. In the space where the so-called Archimedes axiom didn't hold, as is the case in non-standard mathematical analysis (see Robinson [5] pp. 55ff. and pp. 266-7), the answer to the question of whether it is dangerous or not to look at the cube from the right might depend on the nature of the infinitely small portion, or portions, of matter located between the set of 'red' and 'green' parallelepipeds of non-infinitesimal magnitude and the right side of the cube.

nonoverlapping physical contact either with a ‘red’ or with a ‘green’ portion of matter, not with both of them.

Whatever the situation may be, whether the first parallelepiped, observing from the right, is ‘red’ or ‘green’, and however thin it is, there can be enough room within the rest of the body only for a finite number of ‘red’ and ‘green’ parallelepipeds arranged according to the given law: for any non-infinitesimal ε, however small, it holds that there is an *n* such that $(2^n - 1)/2^n > 1 - \varepsilon$, where *n* is finite.

However, it seems that we are also in trouble when we assert that only a finite number of ‘red’ and ‘green’ parallelepipeds of non-infinitesimal magnitude may be located simultaneously within a given cube according to a geometric progression: for any finite *n*, however great, there is a non-infinitesimal ε such that $(2^n - 1)/2^n < 1 - \varepsilon$, and within that ε more edges of ‘red’ and ‘green’ parallelepipeds of non-infinitesimal magnitude may be located.

The problem can be easily generalized, making it independent of the particular shape of the body and the particular converging progression according to which the parts are to be arranged.

II PROPOSALS

Examining the possible answers to the question of the greatest possible number of non-infinitesimal physically distinguished parts of a limited body, we encountered difficulties both with 'Infinite!' as well as with 'Finite!' as the answer, but I believe that the first of these two answers can be made consistent only under quite *ad hoc* and extremely counterintuitive constraints, whereas the second one can be shown to be not only plausible but also reconcilable with an intuitively acceptable interpretation of standard mathematical analysis.

To make the infinitist answer[3] consistent, we have to allow for the possibility of the existence of a limited body consisting of an infinite number of non-infinitesimal physically distinguished parts homogeneous in themselves without a first non-infinitesimal homogeneous part at a side of that body, which either entails very bizarre consequences or requires the. postulation of the non-meaningfulness of otherwise permissible and intuitively plausible questions.

Thus, recalling our example in Section I, we should either have to admit that by looking from the right at the body that consists of only 'red' and 'green' parallelepipeds of non-infinitesimal magnitude we can see, even in principle, neither a 'red' nor a 'green' portion of matter, every 'red' part being screened by a 'green' one and every 'green' part being screened by a 'red' one, or we should have to postulate that the question of what we see, looking at that body from the right, is simply nonsensical.[4]

Similarly, we should have to assert that it is not possible to come in contact with such a body on its right side without penetrating it, despite the fact that we have adopted the possibility of non-overlapping physical contact by the very description of the arrangement of 'red' and 'green' portions of matter which constitute the body.

It is easy to cite many other natural and common questions which require bizarre answers or must be made nonsensical by definitional *fiat* if the infinitist thesis is to be consistent. The answer that the number of 'red' and 'green' parallelepipeds of non-infinitesimal sides simultaneously located within a given cube is necessarily finite involves no *ad hoc* and counterintuitive constraints.

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[3] For the elaborated versions of infinitism, see Russell [6] lectures VI and VII, Grünbaum [3] ch. Ill, and Salmon [7] ch. II.

[4] For the elucidation of this point I am indebted to my friend Timothy Williamson.

The fact that there is no greatest number of parallelepipeds which may be located within a cube according to a geometric progression can hardly prove, by itself, that there is enough room for an infinite number of them positioned there simultaneously (see Section III below), and if we have an independent justification, as is the case, to assert that there is not enough room for an infinite number of them located there simultaneously, the only thing to do is to offer an interpretation which reconciles the two facts: the fact that there is no greatest finite number of 'red' and 'green' parallelepipeds of non-infinitesimal sides which could be located simultaneously within a given cube according to a geometric progression and the fact that an infinite number of them cannot be located there simultaneously.

The two facts can be reconciled by simply adopting the assumption that the number of physically differing parts *can* always be greater than it is in *a particular case*, whereas the number of them *must* be finite in *any particular case*.

The two theorems of standard analysis, $\forall n \exists \varepsilon ((2^n - 1)/2^n < 1 - \varepsilon)$ and $\forall \varepsilon \exists n ((2^n - 1)/2^n > 1 - \varepsilon)$, which we encountered in Section I, may be interpreted as supporting the first and the second part of this assumption respectively.

The first of the two theorems obviously suggests that there is always enough room for more physically distinguished parts to be arranged in accordance with the same geometric progression that governs the arrangement of a finite number of physically distinguished parts already present within a unit cube in a particular case. The second one suggests, however, that in no particular case can an infinite number of such parts be located within a unit cube in such a way, insofar as it shows that for any possible point of the [0, 1) interval it holds that only a finite number of edges of the parts of a body may be located proceeding from zero to that point according to that law; the addition of one single point, of point 1, can hardly make the unit interval roomier, [0, 1] being metrically equal to [0, I).[5] Consequently, there is always at least one of the physically distinguished parts of a given cube which is not positioned in accordance with an infinite mathematical series.

III DISCUSSION

It may seem that the proposal favoured in the previous section can be shown to be implausible by the following reasoning.

Let us imagine a cube consisting of 'red', 'green' and 'blue' stuffs (where 'red' and 'green' are to be understood as before, and 'blue' analogously). According to our assumption above, in spite of the fact that the number of 'red' and 'green' parallelepipeds located within a cube is necessarily finite in any particular case, if they are arranged according to a geometric progression there must be enough room within the rest — i.e. within the 'blue' parallelepiped in the present case — for more physically distinguished parts to be located in accordance with the same geometric progression according to which the 'red' and 'green' parallelepipeds are arranged. Thus we can introduce 'RED' and 'GREEN' parts of the given cube, denoting not only the 'red' and 'green' parts respectively, but also all those parts of the 'blue' parallelepiped which would be respectively 'red' or 'green' in any of the possible particular cases in which more 'red' and 'green' parallelepipeds of non-infinitesimal magnitude were arranged according to the same law as applied to the 'red' and 'green' parts already present there.

The number of 'RED' and 'GREEN' parallelepipeds cannot be finite, because no finite number of 'RED' and 'GREEN' parallelepipeds exists that exhausts the whole cube and hence

[5] The mathematicians may speak correctly of an infinite number of (nonoverlapping) parts of a body not only because they speak of no real cases, but also because they speak of no particular case at all, be it real or possible, so that the number of parts is not fixed and can be always said to be greater than any given finite number (cf. below, Sections III, IV).

for any finite number it holds that more 'red' and 'green' parallelepipeds could be located within the rest of the cube, more 'RED' and 'GREEN' parts being, consequently, already present within the cube. If the number of 'RED' and 'GREEN' parallelepipeds already present can be no finite number, the number of these parallelepipeds must be infinite. And if any of an infinite number of 'RED' and 'GREEN' parallelepipeds, already present within the cube, can be respectively 'red' or 'green' it seems that we must accept that the number of 'red' and 'green' parallelepipeds simultaneously located therein can be infinite too.

However, this argument either begs the question or it is simply not conclusive.

In order to justify the claim that the number of 'red' and 'green' parallelepipeds can be infinite because the number of 'RED' and 'GREEN' parallelepipeds is infinite, one would have to assert not only that any of the 'RED' and 'GREEN' parallelepipeds can be 'red' or 'green', but also that an *infinite* number of 'RED' and 'GREEN' parallelepipeds can become respectively 'red' or 'green' in a *particular case*.

'RED' and 'GREEN' parts are those parallelepipeds which already are, or could be, 'red' or 'green' in any of the possible particular cases. If one asserts that the number of 'RED' and 'GREEN' parallelepipeds is infinite because within the set of all possible particular cases there is an infinite sequence of 'red' and 'green' parallelepipeds, then the assumed is what is to be concluded (that an infinite number of 'red' and 'green' parallelepipeds can be simultaneously located within the cube). If, however, the assertion that the number of 'RED' and 'GREEN' parallelepipeds is infinite is based only on the fact that there is no greatest number of 'red' and 'green' parallelepipeds which can be located within a cube, this is too weak to justify the claim that an infinite number of 'RED' and 'GREEN' parallelepipeds can become 'red' or 'green' in a particular case — which can easily be seen with the aid of the following, sufficiently analogous, example.

Let us suppose that two immortals decide to have many children. There is no greatest number of children they can have, hence we may suppose that they have decided to put no limits on the number of children they will have: every year, another child. Let us introduce CHILD to denote not only any of the children the two immortals have, but also any of the future children. Now, the number of CHILDREN is not finite and any given member of the infinite set of CHILDREN can, and even will, become real as well as any given finite subset of the set of CHILDREN, an infinite number of CHILDREN never becoming real.[6]

Analogously, we may continue to assert that there is no greatest number of 'red' and 'green' parallelepipeds which can be located and 'GREEN' parallelepipeds is infinite, but also — again analogously, though not for the same reasons — that it is impossible for an infinite number of 'red' and 'green' parallelepipeds ever to be positioned within the cube, some of the 'RED' and 'GREEN' parts of the cube, i.e. an infinite number of them, being thus necessarily not as real as are those which are 'red' and 'green'.

IV CONCLUSION

Finding the reasons supporting the impossibility of an infinite number of non-infinitesimal physically distinguished parts present within a limited body convincing enough, while finding, at the same time, the argument against the reconciliation of the statement about the necessity of the finiteness of the number of such parts and the fact that their number can always be greater than it is in a particular case inconclusive, I subscribe to the assertion that an infinite number of parts of a body which are referred to in a purely mathematical way are not as real as physically distinguished parts.

[6] This kind of infinity was called by Cantor the 'improper infinity' (*die uneigentliche Unendlichkeit*); [2] p. 165.

In its consequence, this result sides with Aristotle's teaching on what is to be considered as a real unit in the physical world.[7] Though oneness, multitude, and existence can be spoken of in multiple ways, there are certain limitations which have to be put on the widespread anti-realist view of the number of parts of a thing. It is true that questions such as 'How many things are there in this room?' and 'How many parts does this thing consist of?' are ill-formulated, because 'thing' and 'part' are not sufficiently specified sortals for counting entities. It is also true that we may choose different sortals, obtaining thus different counts, and to that extent the question of whether something is one single thing or a set consisting of more or fewer elements is really a matter of linguistic convention.[8] But this is so only up to a certain point if it is true that a limited body can consist of only a finite number of physically distinguished parts of non-infinitesimal magnitude.

If a limited body, even in principle, can contain, and consist of, only a finite number of such parts and if — because of that — an infinite number of parts singled out in a purely mathematical way cannot be said to be as real as physically differing parts, then there is an ontologically relevant sense in which it may be said that there are real — though not indivisible — units, a given body containing, and consisting of, a necessarily finite — though not *a priori* fixed — number of such units: physically distinguished parts homogeneous and continuous in themselves.

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[7] If there are no indivisibles, the real unit is, primarily, that which is homogeneous and continuous in itself, because it is only potentially multiple (cf. Aristotle, *Physics*, 185b9-196a3, 227b20ff., 236b7).

[8] Cf. Frege, *Grundlagen der Arithmetik*, §30.