The Philosophical Impact of the Löwenheim-Skolem Theorem

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Abstract Not only that the theorem which Löwenheim proved 1915 was the first big result in what we now call Model Theory, but, primarily due to Skolem, who profoundly analyzed and understood the significance of its far-reaching consequences, the Löwenheim-Skolem Theorem made also a revolutionary impact on the history of the twentieth century mathematics, and philosophy of mathematics in particular. Among the consequences, the most disastrous were those that concerned Hilbert's categoricity demand and Cantor's concept of cardinality. In this article, it is argued that though it should be admitted that the first group of consequences, related to the possibility of non-standard models, clearly pointed to the expressive weakness of the (first-order) language in which, in the first three decades of the last century, the main mathematical theories were expected to be formalized, this lesson concerning language was only the first part of the story. The need of re-investigation of the concept of relational structure, and the concept of cardinality in particular, became acute only in view of results of Paul Cohen, Solomon Feferman and Azriel Lévy in the seventh decade of the century. It is shown how the relativity of cardinality is to be understood and why, instead of being attributed to sets as such, it should be rather attributed to sets as basic sets of relational structures. It is also shown that, if properly understood, the relativity of cardinals may be relevant not only for the philosophy of mathematics but for metaphysics as well.

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1 The historical philosophico-mathematical background within which the Löwenheim-Skolem Theorem appeared

1.1 The Relation Externalism

In his Intellectual Biography [30, Ch. V], Bertrand Russell named the year 1889 as a turn, which he relates not only to his own philosophical development but also to the beginning of an era, in which, contrary to the opinion of Leibniz and Hegel, relations between objects were started to be treated as equally real as and not reducible to the properties of objects between they hold. For instance, if *a* and *b* are two objects, then *a* and *b* plus a relation holding between them make up a relational structure that is equally real as *a* and *b* themselves, and in addition—and this is a consequence particularly important for our present purposes—objects *a* and *b* would remain the very same objects if they ceased to stand in a given relation and started to be in a different relation.

1.2 Referring to objects, properties and relations according to the theory of meaning holism: Frege, Wittgenstein and Hilbert

Frege was the first who explicitly introduced the theory of meaning holism by stating that a concept is something unsaturated [13, p. 24] [16, I.1. pp 33–34], which can become a reference only by being ascribed to an object. So, for instance, if the concept horse occupies the place of the grammatical subject in a sentence, it does not function as a concept [14].

As for the way in which we refer to objects, Frege applied the difference between sense and reference (*Sinn* and *Bedeutung*) in order to show that we normally, if not always, refer to an object through some mode of its presentation [15, pp. 57, 62, 67]. So, for instance, 'the Morning Star' and 'the Evening Star' are two different names due to the fact that they represent two different ways in which we refer to Venus, but they have the same reference, since they both refer to one and the same planet. One might raise the question about the name 'Venus', since it seems that, contrary to 'the Morning Star' and 'the Evening Star', 'Venus' refers directly to Venus. It is perhaps possible to say that 'Venus' is actually a shorthand for, say, 'the second planet in the Solar system', being as such also a mode of presentation of the very same object to which 'the Morning Star' and 'the Evening Star' refer. But there are cases in which there is nothing like 'Venus' in the given example. 'The centre of the circle inscribed in an equilateral triangle' and 'the centre of the circle circumscribed about the same equilateral triangle' are two different ways of referring to one and the same point, but there is no name that would refer directly to it. And if we refer to this point by 'the centre of gravity', it is clear that this is just another mode of presentation of the same object which 'the centre of the circle inscribed in an equilateral triangle' and 'the centre of the circle circumscribed about the same equilateral triangle' refer to.

Wittgenstein radicalized the theory of meaning holism by stating that names have no reference at all outside sentences in which "state of affairs" are stated [40, 3.3], for the world is, according to Wittgenstein, "the totality of facts, not of things" [40, 1.1].

And finally, it was only Hilbert who stated that there is no reference, either in view of names (constants or, indirectly, individual variables) or of relations (relation constants) or of the statements (about relations holding between the objects which the constants refer to or the variables range over) outside a whole formal theory. So, according to Hilbert, it is only a whole system of axioms (a formal theory) that implicitly defines objects and relations which the theory is about [20, 21]. Consequently, the whole formal theory gets its reference through a simultaneous interpretation of its all basic symbols and well-formed formulae, and this reference is a relational structure in which the theorems of the theory are satisfied, i.e. true.

1.3 Consistency, Completeness and Categoricalness

According to Hilbert's Programme, the ideal formal system should be consistent, complete and categorical [12, Ch. V, §4].

Syntactically, an axiom system is consistent if an only if there is no formula *A* of the system for which both *A* and its negation can be proved. Semantically, a system is consistent if there is an interpretation such that its axioms are true. As a consequence, later formulated as a theorem of the Model Theory, an axiom system (a formal theory) is consistent if and only if it has a model. So, in order to prove that a system is consistent, it would be sufficient to show that it has a model. However, for doing this, one should have a well-established meta-theory concerning the existence of a model. That's why Hilbert wanted to have a purely formal proof of the consistency without relating it to a model. However, in many interesting cases such a proof is not of elementary nature and requires always stronger and stronger theories, as it follows from Gödel's Second Incompleteness Theorem [18]. This problem of Hilbert's Programme is, however, irrelevant for our main concern.

The question concerning syntactical completeness is indirectly relevant to our topic. A formal system is syntactically complete if and only if there is no pair of sentences—a sentence and its negation—such that neither of the two sentences is a theorem of the system.

The demand concerning the categoricalness of a system will be of crucial importance for our main concern. An axiom system (a formal theory) is categorical if and only if all its models are isomorphic, i.e. if and only if all the relational structures in which the system is interpretable are such that there is a structure preserving one-one mapping between the elements of their basic sets.

1.4 The set equipotency and higher-order infinities

One of the central things that are going to be questioned by the Löwenheim-Skolem Theorem concerns Cantor's theory of higher-order infinities, which is based on the conception of the power of a set [12, pp. 95ff].

Two finite sets are equipotent if and only if all the elements of one of them can be brought into $1-1$ correspondence with all the elements of the other one. That's why, for instance, the set of four people has the same power as the set of four apples. By generalizing this idea, two infinite sets are also said to be equipotent if and only all if the elements of one of them can be brought into $1-1$ correspondence with all the elements of the other one. Now, a set has less power than some other set if and only if all its elements can be brought into $1-1$ correspondence with the elements of some proper subset of the latter set but, at the same time, there are not enough elements of the former with which all the elements of the latter could be brought into $1-1$ correspondence. That's why a set of five apples has a greater power than a set of four people. And that's also why, according to several proofs that Cantor has offered, the set of natural numbers is equipotent with the set of rational numbers but has a less power than the set of real numbers. Cardinal numbers are numbers that denote set powers. So, $\aleph_0$ denotes the power of all the infinite sets that are equipotent with the set of natural numbers, and these sets are the weakest infinite sets. Cardinal numbers $\aleph_1, \aleph_2, \aleph_3, \ldots$ denote infinite sets of always greater and greater power. Cantor had stated and believed to have proven [5, p. 333] that the power of the set of real numbers is $\aleph_1$, namely, that there is no infinite sets whose power would be greater than the power of the set of natural numbers but lesser than the power of the set of real numbers, but since his proof has turned out to be inconclusive, this statement of his was later called the Continuum Hypothesis.

1.5 The set orderings

In order to get a relational structure, we have to start with a basic set and then define at least one relation on it. The most important relation is the ordering relation, i.e. either $<$ or $\leq$, where the latter can be defined via the former and the identity relation. However, it is important to notice that, by starting with different basic sets, we can get different orderings—in accordance with the Cantorian procedure—by using the allegedly same ordering relation. So, for instance, the three relational structures, $\langle \mathbb{N}, \leq \rangle$, $\langle \mathbb{Q}, \leq \rangle$, $\langle \mathbb{R}, \leq \rangle$, where $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ are the set of natural numbers, the set of rational numbers and the set of real numbers, respectively, are not ordered in the same way: the first structure is discrete, the second one is dense but not continuous, and the third one is continuous.

Concerning the order of $\langle \mathbb{N}, \leq \rangle$, it will be very important, for the understanding of an apparent paradox of the Löwenheim-Skolem Theorem, that, in the meta-theory, the ordering relation of the standard model can be introduced in two seemingly equivalent ways, whose difference, however, can be exemplified by using a model of Non-Standard Arithmetic. Namely, since the well-order of the intended model is well-founded and total, it seems that any of the following two pairs of conditions is sufficient for its definition. We can either stipulate that (1) there is an element which is the minimal element of the structure; and that (2) for any element, there is a unique element that is his immediate successor, or, alternately, we can retain the first condition and add: (2') any non-empty subset of the basic set has a unique minimal element. But, as we shall see below (see 3.2), there are structures in which conditions (1) and (2) are satisfied, whereas (2') is not. This means that (1) and (2) are not sufficient for defining, in a categorical way, the order of natural numbers in Standard Arithmetic, whereas (1) and (2') are.

As for the difference between $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$, it consists only in the fact—which Cantor held to be the first who had discovered and defined it clearly [5, p. 190]—that while each element of $\langle \mathbb{Q}, \leq \rangle$ is an accumulation point of an infinite number of elements, it is true only in $\langle \mathbb{R}, \leq \rangle$ that each accumulation of an infinite number of elements has as its accumulation point the element that is an element of the basic set itself. The lesson, which will be very important for an interpretation of the Löwenheim-Skolem Theorem, is that the allegedly same ordering relation functions differently in view of $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, so that the difference between structures which are discrete, dense but not continuous, and continuous depends on how they are structured partly independently on how they are ordered according to the $\leq$ relation. In particular, if a structure is well-ordered according to the second definition, its basic set is countable, but it is a question, as we shall see below, whether it is so in the case in which a structure is well-ordered according to the first definition only.

And finally, from the Hilbertian point of view, the difference between discrete, dense but not continuous, and continuous structures is to be obtained only through different axioms defining implicitly the meaning of the $\leq$ relation. This means that, according to the Hilbertian meaning holism, the $\leq$ relation cannot be said to be the same in each of the three cases. More concretely, though it might seem that the same ordering relation holds between elements of the otherwise differently structured elements of discrete, dense but not continuous, and continuous structures, the very difference between these structures is based on the fact that the ordering relation is implicitly defined in three different ways relating to these three structures.

1.6 Intuitionism and Platonism

Though Intuitionists were not directly involved in the discussion concerning the Löwenheim-Skolem Theorem due to the fact that the Intuitionist Programme remained untouched either by the Theorem itself or by its consequences, the concept of constructability [12, pp. 61, 104, 108], though not in the sense in which Intuitionists understand it, will appear in some important examples concerning the problem of changing cardinalities [25].

Intuitionists reject the Cantorian concept of the actual infinity and use only the Aristotelian concept of the dynamic (or potential) infinite. So they always start with a finite number of constructed objects that may then get greater and greater unboundedly but never becomes actually infinite [4, pp. 270ff.]. This treatment enables them to prove various statements about the objects introduced in such a way whenever they have a recursive control over what they speak about and an inductive way to prove a theorem, but they do not allow us to speak of infinite sets or classes as ‘finished entities’ [4, p. 433] but only of “spreads” as entities “in statu nascendi” [39, p. 52]. For instance, we may speak of natural numbers as a species, without restricting our discourse to a finite number of them, and can also prove, by using mathematical induction, that any of these numbers must be odd or even, but we mustn’t speak of “the set of all the natural numbers (whose cardinal number is $\aleph_0$).”

An important consequence is that we cannot use the Weierstrassian concept of real numbers according to which any complete decimal expansion defines a unique real number. We may say, for instance, that 0,33..., where 3 is supposedly going to occur at any place of the decimal expansion, defines the unique number, i.e. $\frac{1}{3}$, but we mustn’t take that the decimal expansion of π defines π as the unique real number, because, firstly, there is no recursive way according to which such an expansion would be defined, and, secondly, there is no mathematical object such as the complete infinite decimal expansion.

Due to the given restriction under which one is allowed to speak of the existence of mathematical objects and to prove the existence of their properties and relations holding between them, the use of many classical logical principles and derivation rules is also to be restricted. It is so in the case of the principle of excluded middle, the counterposition, the double negation, the reductio ad absurdum and so on. In particular, this prevents all the Cantorian proofs concerning the existence of various types of infinity and the uncountability of the set of real numbers.

Now, I shall call Platonists all those who do not accept the intuitionist rigors concerning the existence of mathematical objects and their way in which the theorems are only allowed to be proved. This means that Platonism will be taken in a much broader sense than as denotation of the mathematical programme contrasted to Logicism, Intuitionism and Formalism. In particular, Hilbert’s Programme will not be contrasted to Platonism, since the rigor of Hilbert’s foundation of mathematics concerns the syntactical finitism and recursive control that should govern the introduction of basic symbols and formation and derivation rules [23, pp. 137ff], which is, as such, not directed against the transfinite mathematics that lies on the semantical part of a formal theory. As Hilbert put it himself, “No one shall drive us out of the paradise which Cantor has created for us” [23, p. 141].

2 The Löwenheim-Skolem Theorem and its generalization

2.1 What is the Löwenheim-Skolem Theorem about?

The essential statement of what is now called the Löwenheim-Skolem Theorem as well as its proof—in spite of some errors and slopps [38, p. 156]—are to be found in Löwenheim's famous paper [26]. However, due to the fact that Skolem, in his four papers [31, 32, 34, 35], removed all gaps and omissions from Löwenheim's proof, got rid of any use of the Axiom of Choice, strengthened and extended the Theorem, analyzed profoundly its meaning, formulated the Theorem related Paradox and offered its first resolution, his name was later rightly attached to the name of Löwenheim when referring to the Theorem itself. Some go even further on and call it the Skolem-Löwenheim Theorem [12, p. 302].

The Theorem is nowadays highly estimated as the first great result in what was later called the Model Theory [38, p. 154], viz. as a contribution that proved something substantially important and seemingly paradoxical about the relation between a formal theory and its interpretation.

Since, perhaps contrary to, say, Gödel's Incompleteness Theorem, the understanding of the Löwenheim-Skolem Theorem and its consequences represents a problem per se that does not depend essentially on the understanding of its proof, we shall turn directly, after giving its main formulation, to the clarification of its meaning.

2.2 The main formulation of the Löwenheim-Skolem Theorem and the straightforward meaning of its strong version

Löwenheim's original formulation [26] was about a first-order sentence σ that has a model. However, since σ can be a conjunction, we may speak, instead of σ, of a set Σ of first-order sentences. In particular, Σ can be the set of axioms of a formal theory. As for the model, since no qualification is indicated, the basic set of the relational structure the Theorem is about can be an infinite set of any cardinality whatsoever.

Theorem 1. *If Σ has a model whose basic set is infinite, then Σ has a model whose basic set is countable.*

Let us explain the straightforward meaning of the Strong Version of the Theorem, where, given that A is the basic set of the original structure and B the basic set of a countable model B, $B \subseteq A$. Let S be a structure which consists of (1) an infinite set A whose cardinal $|A|$ is greater than $\aleph_0$, and (2) a finite or denumerable number of relations $R_1, R_2, \ldots$ defined on it. Then, there is a structure S' that consists of (1) the basic set B whose cardinal is $\aleph_0$, and of (2) the relations $R'_1, R'_2, \ldots$ that are just the relations $R_1, R_2, \ldots$ of S restricted to the set B, so that for every sentence σ of the first-order language which corresponds to S, i.e., whose extra-logical symbols are just relation-symbols which refer to the relations $R_1, R_2, \ldots$ of S, σ is true in S' if and only if σ is true in S. In particular, if **T** is a first-order theory and S is a model of **T**, then S', too, is a model of **T**.

2.3 The generalized versions of the Löwenheim-Skolem Theorem

In 1928, Tarski presented in his seminar a form of what is now called the Upward Löwenheim-Skolem Theorem [38, p. 160]. However, this result was never published and was only mentioned in the editor's note of Skolem's paper that appeared six years later [36]. So, the proof is to be found only in the famous paper of Malcev [27].

Theorem 2. *If Σ is countable and has a model whose basic set is infinite, then Σ has a model in each infinite power greater than the power of the original basic set.*

Since the Theorem can be generalized so as to state that Σ has a model in each infinite power lesser than the power of the original basic set—which is its version called the Downward Löwenheim-Skolem Theorem—the most generalized form states that:

Theorem 3. *If Σ is countable and has a model whose basic set is infinite, then Σ has a model in each infinite power.*

However, since, from a philosophical point of view, the original version of the Theorem is sufficient for the formulation and understanding of the most intriguing questions and the most interesting examples related to its consequences, we shall in what follows focus our attention to this form of the Theorem and refer to it, unless necessary, by using its name without qualification.

3 The far-reaching consequences of the Löwenheim-Skolem Theorem

3.1 The general problem concerning Hilbert's Programme caused by the Löwenheim-Skolem Theorem

As it is said above (2.1.), the Löwenheim-Skolem Theorem concerns the relation between a first-order formal theory and its interpretation. Now, we have to remember that at the time at which the Löwenheim Theorem appeared one of the main concern of mathematicians was to formulate, as first-order formal theories, the set theory, the theory of elementary arithmetic, the theory of real numbers and of the continuum in general (within the formalized set theory or independently of it). These theories were expected to be in accordance with Hilbert's Programme, i.e., to be consistent, complete and categorical, expressing formally and unequivocally all the truths discovered informally or semi-formally in the respective mathematical theories, i.e., in the Cantorian set theory, the Fregean or Dedekindian Arithmetic and in the general theory of the continuum applicable in the theory of real numbers as well as in geometry. For this "paradise state," in which a real breakthrough concerning the formal foundation of the most important mathematical theories was expected, the Löwenheim's Theorem and Skolem's analysis of its consequences represented a real disaster.

In the first place, the Löwenheim-Skolem Theorem implies the existence of non-intended models of all the mentioned formal theories that are non-isomorphic with the intended models. This means that these theories are necessarily not categorical.

The non-categoricity of a theory means that we cannot formally distinguish what is distinguishable in the corresponding informal or semi-formal theory. In particular, the cardinality becomes something relative [11, pp. 108ff], for, according to the generalized version of the Löwenheim-Skolem Theorem, the structure in which the theory is interpreted can be taken to be of any cardinality whatsoever. This relativeness of the cardinals was very disturbing both to Skolem and von Neumann [32, pp. 223ff] [28, pp. 239–240]. For von Neumann, it suggests a kind of unreality of cardinals and therefore serves as the argument in favour of Intuitionism.

And finally, one can make one step more, which Skolem did, and raise the question about a possible paradox [33], since, as we shall see, a sentence stating the existence of uncountable sets can be true after being interpreted in a structure whose basic set is denumerable.

3.2 The non-categoricalness and the formal indistinguishability of the informally distinguishable

In order to illustrate the problem of the non-categoricity of a formal theory, I shall start with the formal theory of elementary arithmetic as the simplest and most obvious case.

The objects of the basic set of the relational-operational structure that is the intended model of the fully formalized elementary arithmetic are numbers $0, 1, 2, 3, \ldots$ for which the so-called Archimedes Axiom holds and which are all finite in spite of the fact that there is an infinite number of them. As it is standardly defined, "an Archimedean model of arithmetic is a model in which for every number N and for every [positive] number ε there is a finite number n such that $\varepsilon + \varepsilon + ... + \varepsilon > N$, where ε is taken n times" [1, pp. 926f].

Let us imagine, however, a structure that does not differ from the intended model in any other respect except that in it the Archimedes Axiom does not hold. This means that in the basic set of this structure, in addition to finite numbers, there are numbers that are infinite in the sense that they cannot be reached in a finite number of steps by starting from any number that is finite in the sense in which all members of the basic set of an Archimedean model are finite. It is evident that this non-Archimedean structure is not isomorphic with the structure that is the intended model of the formal theory of elementary arithmetic.

However, though in such a non-Archimedean model there are numbers a and b such that there is no n that is finite and such that $a \times n \geq b$, the structure supposedly does not differ from an Archimedean structure in no other respect. So, every number, be it finite or infinite, has its unique immediate successor.

Now, given that there is an infinite number of infinite numbers just as there is an infinite number of finite numbers, the fact that there is no maximal element of the set of finite numbers in the given non-Archimedean structure is completely analogues to the fact that there is no minimal element of the set of infinite numbers. This means, in effect, that there is an infinite subset of the set of elements of the basic set of the given non-Archimedean structure that does not have a minimal element, which means, consequently, that, in view of two pairs of conditions cited in 1.5, conditions (1) and (2) are satisfied, whereas the condition (2') is not.

One would certainly like that, in accordance with Hilbert's programme, the formal theory of elementary arithmetic grasps the difference between the Archimedean and the Non-Archimedean arithmetic so that if the former is a model of the formal theory, the later is not. Unfortunately, this cannot be done if the formal theory is a first-order theory.

The problem is that the axioms that would be formulated in the style of Dedekind and Peano do not enable us to preclude the existence of infinite (or hyper-finite) numbers in the basic set of the intended model of the theory. For instance, it will be true that for any two a and b such that $0 < a$ and $a < b$, it holds that there is n such that $a \times n \geq b$, independently on whether we interpret the formal theory in an Archimedean or in a Non-Archimedean structure, for if b is a hyper-finite number, n can be a hyper-finite number as well. One could try to impose a limitation to standard numbers, as Fraenkel did in a similar context [10, pp. 233–234], by adding the Axiom of Restriction to Peano's arithmetic, say of the form

$$\forall x[x = 0 \vee x = 0' \vee x = 0'' \vee \ldots],$$

but this expression is of infinite length and is, therefore, not legitimate within the framework of the standard first-order theory. And there is no way of reformulating it within the standard framework.

If one tries to use the fact that in the non-Archimedean structures there are infinite sets that have no minimal element whereas in the Archimedean structures it is not so in order to make the difference between the two, this would also lead nowhere, since the variables of the formal theory of arithmetic range over the numbers and not over their sets. To express the difference, one would need a second-order language.

All in all, there is no way to avoid the non-categoricalness by interpreting the first-order formal theory of arithmetic in such a way that it becomes also possible to distinguish formally what is distinguishable in the meta-theory.

3.3 Skolem's Paradox

It was only in 1928 that Hilbert and Ackermann formulated quite precisely the concept of the completeness of a logical syntax with respect to a given semantic theory [22]. Only two years later, Gödel was quick to prove, in his doctoral dissertation, his famous Completeness Theorem [17], which is now called simply Gödel's Completeness Theorem, in contrast to his Incompleteness Theorem [19], which concerns the question of the syntactical incompleteness (see 1.3 above). Gödel's Completeness Theorem says that:

Theorem 4. *A sentence σ of a first-order formal theory Σ is true in all the models of Σ if σ is a theorem of Σ.*

By using this theorem, it is easier to reach the point of Skolem's Paradox than by the way in which it was done by Skolem himself, who, for this purpose, could use only the Löwenheim-Skolem Theorem itself.

By Gödel's Completeness Theorem, if a first-order formulation of the Zermelo-Fraenkel Set Theory (**ZF**) is consistent, then each theorem of **ZF** is true in any of its models. Now, by the Löwenheim-Skolem Theorem, one of the models of **ZF** is denumerable. Let S be such a model and let A be the basic set of S, and R a binary relation formulated in **ZF** and interpreted in S as $\in$ defined on A. For the sake of convenience, let us take that, if a member c of A stands in the relation R to a member b of A, c is a member of b in S, i.e., we shall speak of b as if it were the set $\{c \mid cRb\}$ (of all the members c such that cRb). In addition, let us denote by ω the only member x of A that satisfies in S the formula 'x is the least infinite ordinal.' Now, on the one hand, it is a theorem of **ZF** that there are uncountable sets, so, (1) the set A must have a member a such that it is true in S that a is not denumerable. On the other hand, however, (2) all the members b of a in S are members of A, which is supposedly a denumerable set. (1) and (2) are seemingly inconsistent, and this is what is known as Skolem's Paradox [12, p. 303].

Since, as mentioned above, (1) and (2), taken together, seem to imply the relativization of the cardinality, Skolem's Paradox is sometimes also understood as referring to this fact, if, namely, one is prone to believe, as von Neumann was, that the relativeness of cardinality is inconsistent with the way in which the very concept of cardinality is to be understood.

4 The positive reactions to the Löwenheim-Skolem Theorem: to blame the language or to re-investigate structures?

Confronted with all the unpleasant consequences of the Löwenheim-Skolem Theorem and of Skolem's Paradox in particular, one can try, in view of the fact that the Theorem concerns the relation between a formal theory and its interpretation, to find one of the following two ways out of the situation: to blame the first-orderness of the language in which formal theories are supposedly formulated and use a stronger language to formulate them or to re-investigate the very structures the theories are about in order to re-define at least some of the key concepts underlying their understanding. Or perhaps, as the third possibility, one can find that it is necessary to do both.

4.1 The weakness of the language

As a consequence of the so-called linguistic turn in philosophy that happened at the beginning of twentieth century, one could reasonably expect that the first option was ready to be endorsed, both by mathematicians as well as by philosophers. A general lesson of the linguistic turn has been that very many, even if not all, problems and apparent paradoxes which we are confronted with by dealing with reality have their origin in the language we use to speak of it. Isn't it so also in the philosophy of mathematics, where, at those days, the stubborn practice to stick to first-order theories was nearly canonized?

The idea of blaming the language of formal theories for the disastrous consequences of the Löwenheim-Skolem Theorem may arise quite naturally by analysing the informal or semi-formal theories themselves which the main formal theories were to formalize. The point is that the axiom systems used by mathematicians were formulated within theories—such as informal set theory in the first place—which were essentially second-order theories, so that one had indeed categorical axiom systems for natural numbers theory, for real numbers theory and for geometry. It is a bit strange that this fact had not been earlier anticipated as a possible source of the problem that later emerged as a consequence of the Löwenheim-Skolem Theorem. Let me give an example.

As mentioned in 1.5, Cantor was the first who clearly realized that there are two conditions which have to be met if a structure is to be continuous. A set of elements makes up a continuum if and only if (1) it is perfect, and (2) coherent (*zusammenhängend*) [5, p. 190]. The first condition is easy to formulate within a first-order language, because the density axiom, if added to the rest of axioms defining a linearly ordered structure, implies that in any model there is an infinite number of elements accumulating about any of its elements. So, $\langle \mathbb{Q}, \leq \rangle$ as the standard perfect structure, is implicitly defined by the following eight axioms:

(1) $\forall \alpha_n \neg \alpha_n < \alpha_n$
(2) $\forall \alpha_l \forall \alpha_m \forall \alpha_n ((\alpha_l < \alpha_m \wedge \alpha_m < \alpha_n) \rightarrow \alpha_l < \alpha_n)$
(3) $\forall \alpha_m \forall \alpha_n (\alpha_m < \alpha_n \vee \alpha_n < \alpha_m \vee \alpha_m = \alpha_n)$
(4) $\forall \alpha_l \forall \alpha_m \forall \alpha_n ((\alpha_l = \alpha_m \wedge \alpha_l < \alpha_n) \rightarrow \alpha_m < \alpha_n)$
(5) $\forall \alpha_l \forall \alpha_m \forall \alpha_n ((\alpha_l = \alpha_m \wedge \alpha_n < \alpha_l) \rightarrow \alpha_n < \alpha_m)$
(6) $\forall \alpha_m \exists \alpha_n \; \alpha_m < \alpha_n$
(7) $\forall \alpha_m \exists \alpha_n \; \alpha_n < \alpha_m$
(8) $\forall \alpha_m \forall \alpha_n (\alpha_m < \alpha_n \rightarrow \exists \alpha_l (\alpha_m < \alpha_l \wedge \alpha_l < \alpha_n))$

where the last axiom is the density axiom.

However, the second condition cannot be formulated within a standard first-order theory. Namely, in order to say that the basic set is not only perfect but also coherent (*zusammenhängend*), we have to mention explicitly an infinite number of elements, for, according to Cantor, a set is coherent only if any accumulation of an infinite number of elements has the accumulation point that is an element of the basic set itself. In other words, any infinite accumulation from the left to the right must have the least upper bound that is an element of the basic set itself just as any infinite accumulation from the right to the left must have the greatest lower bound that is an element of the basic set itself. This can be expressed only in the second-order language or in the extended first-order language, which is now known as the infinitary language $L_{\omega_1\omega_1}$. For the reasons that will be mentioned below (in 5.4), let me formulate the two necessary axioms in the language $L_{\omega_1\omega_1}$ [2]:

(9) $$\forall\alpha_1\forall\alpha_2\ldots\forall\alpha_i\ldots (\exists\beta_1 \bigwedge_{1\le i<\omega} \alpha_i < \beta_1 \Rightarrow \exists\gamma_1(\bigwedge_{1\le i<\omega} \alpha_i < \gamma_1 \wedge \neg\exists\delta_1(\bigwedge_{1\le i<\omega} \alpha_i < \delta_1 \wedge \delta_1 < \gamma_1))$$

(10) $$\forall\alpha_1\forall\alpha_2\ldots\forall\alpha_i\ldots (\exists\beta_1 \bigwedge_{1\le i<\omega} \alpha_i > \beta_1 \rightarrow \exists\gamma_1(\bigwedge_{1\le i<\omega} \alpha_i > \gamma_1 \wedge \neg\exists\delta_1(\bigwedge_{1\le i<\omega} \alpha_i < \delta_1 \wedge \delta_1 > \gamma_1)))$$

(where $\alpha_m > \alpha_n \leftrightarrow_{def} \alpha_n < \alpha_m$). Notice that the antecedents in these two axioms are unavoidable because in $\langle\mathbb{R},\le\rangle$, which is the intended model of the system representing a linear continuum, there are infinite subsets of $\mathbb{R}$ without an upper and/or a lower bound, so that what we want to say is that if there is an upper (lower) bound at all, there is also a least upper (greatest lower) bound.

But now, though the above axiom system is formulated in the extended first-order language, the last two axioms contain infinite conjunctions, which was not legitimate at the time at which the continuum theory was to be formulated as a formal theory.

Generalizing the point of the previous example and turning to Skolem's Paradox and all the related problems concerning the non-categoricity of formal theories proposed at the time we are speaking about, we can simply say that the language used in the formulation of these theories was blind for making all the differences expressible only in a non-standard language. In particular, this means that, bearing on mind the meaning of Skolem's Paradox, there can be a relation between the elements of the basic set which is uninterpretable as any relation of a given formal theory but which makes the statement of the uncountability of the basic set true, in spite of the fact that all the relations envisaged by the theory are such that they make the very same basic set countable.

So, one might say that the allegedly paradoxical consequences of the Löwenheim-Skolem Theorem represent nothing else but just a striking example of the weakness of the first-order language for describing the structures in which they are interpreted.

Some mathematicians were ready to accept this as the end of the story. So, already in 1930, Zermelo formulated the set theory in the second-order language [41]. More than three decades later, Abraham Robinson offered a second-order formulation of the non-Archimedean arithmetic and real numbers theory, which contain infinite numbers and infinitesimals [29].

The semantics of the second-order logic is far less clear than the semantics of the first-order logic and, in addition, many philosophers are reluctant, for various reasons, to accept the ontological commitments that follow from the ontology of classes, properties and relations implied by the second-order mathematics. But this is not the matter of our concern. So, let us turn directly to the second strategy of dealing with the consequences of the Löwenheim-Skolem Theorem, which assumes that the Theorem tells us something important about the very relational structures in which formal theories are interpreted.

4.2 Changing cardinalities: the relativeness of cardinality as a language-independent property

If the lesson concerning the weakness of the first-order language in view of mathematicians' attempts to use it by trying to formalize informal set theory, natural numbers theory, real numbers theory, etc. were all that the Löwenheim-Skolem Theorem contributed to, it would certainly be a big result which got rid mathematical and philosophical community of a prejudice that characterized the naïve and ill-founded hope of the "paradise state" in the first three decades of twentieth century. But it would not be what it is now believed to be [11, p. 106]—one of the greatest results in the history of the twentieth century mathematics, which threw a new light on some basic concepts of the set theory and the concept of relational structures in general. The meaning of this latter result became clearly visible only much later through some revolutionary results of Paul Cohen, Solomon Feferman and Azriel Lévy in the seventh decade of twentieth century. But, before turning to these results, I shall try to elucidate the main point by analyzing in a more detailed way Skolem's Paradox itself and by using a quite simple example.

Those who, by resolving Skolem's Paradox, stress the weakness of the first-order formalization of informal set theory for distinguishing cardinalities of different models in which the formal theory is interpretable do not have to stop at this defeatist conclusion, and normally they don't. The *explanation* of the blindness of a formal theory consists in the fact that there can be a relation "invisible" by the theory which makes a countable model uncountable or an uncountable model countable.

As for the first possibility, it is sufficient that one reminds the example concerning the second Cantor's condition for the continuity of an ordered structure (see 4.1). Though, in accordance with Hilbert's Programme, the difference between the ordering relation of only dense and continuous structures should be grasped axiomatically (see 1.5 above), and though this can be done by the use of the language $L_{\omega_1\omega_1}$ (as suggested in 4.1), it cannot be done within a standard first-order theory, and it is exactly the formal indistinguishability between these two ordering relations that makes it possible that the theorem of **ZF** about the existence of uncountable sets is true even if the ordering relation of a structure in which the first-order formalization of **ZF** is interpreted makes its basic set denumerable.

The second possibility is more intriguing. How could it be that a supposedly uncountable model becomes countable? – Let us start with quite a simple case. Suppose that we have a formal theory interpretable in a structure that is dense but not continuous. Is this model also countable? The immediate response will be: “Yes, of course! The set of rational numbers is dense, but it is also countable.” However obvious this answer might be, there is a fact that can be easily overlooked, but which is of crucial importance. We know that there are several functions which define mappings of the set of rationals in such a way that the set of images is directly countable. For instance, we can ‘arrange’ the set of positive rational numbers as follows

$$\begin{array}{ccccccccc}
\frac{1}{1} & \rightarrow & \frac{1}{2} & & \frac{1}{3} & \rightarrow & \frac{1}{4} & & \cdots \\
& \swarrow & & \nearrow & & \swarrow & & & \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \cdots & & \\
\downarrow & \nearrow & & \swarrow & & & & & \\
\frac{3}{1} & & \frac{3}{2} & & \cdots & & & & \\
& \swarrow & & & & & & & \\
\frac{4}{1} & & \cdots & & & & & & \\
\downarrow & & & & & & & & \\
\cdots & & & & & & & &
\end{array}$$

and then pick them up following directions indicated by arrows, obtaining the $1-1$ mapping onto the set of natural numbers, which is directly countable. Since knowledge implies truth, the positive answer to the question of countability of rational numbers seems self-evident. But this “self-evidence” may hide the fact that, independently of our knowledge or ignorance, it is yet the case that the model is denumerable only because there are mappings such as the given one. If, counterfactually, we hadn’t known that there are such mappings, it would have been far from evident that it is so.

The point could be considered philosophically perverse if there were no other, much more interesting cases in which the same phenomenon appears. The most interesting one concerns the denumerability of the set of real numbers.

As his most famous result, Cohen proved 1963 that if **ZF** is consistent, it remains consistent when the Axiom of Choice and the Generalized Continuum Hypothesis are added [6, 7]. Using the model of Cohen’s type applied in this proof, Feferman and Lévy proved that, by omitting the Axiom of Choice, “if **ZF** is consistent, it stays consistent after addition of the following axiom: the set of real numbers is a denumerable union of denumerable sets” [9, p. 593].[1] And then, since it can be proved that the set of elements of a denumerable union of denumerable sets is itself denumerable, it follows that, under given assumptions, the set of real numbers is denumerable!

Ironically, the fact that the set of elements of a denumerable union of denumerable sets is itself denumerable can be proved by the very same method that I have just used above for showing that the set of rational numbers is denumerable, and which was originally used by Cantor himself! Namely, let $B_1, B_2, B_3, \ldots$ be members of a denumerable union and $a_{11}, a_{12}, a_{13}, \ldots$ elements of $B_1, a_{21}, a_{22}, a_{23}, \ldots$

[1] See also [8, p. 146] and [24, p. 142]

elements of $B_2, a_{31}, a_{32}, a_{33}, \ldots$ elements of B_3, and so on. Now, by "arranging" the elements of $B_1, B_2, B_3, \ldots$ as follows

$$
\begin{array}{lccccccccc}
B_1 & a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & \cdots \\
 & & \swarrow & & \nearrow & & \swarrow & & \\
B_2 & a_{21} & & a_{22} & & a_{23} & & \cdots & \\
 & \downarrow & \nearrow & & \swarrow & & & & \\
B_3 & a_{31} & & a_{32} & & \cdots & & & \\
 & & \swarrow & & & & & & \\
B_4 & a_{41} & & \cdots & & & & & \\
 & \downarrow & & & & & & & \\
 & \cdots & & & & & & &
\end{array}
$$

and picking them up following directions indicated by arrows, we obtain the $1-1$ mapping onto the set of natural numbers, which is directly countable.

Further elucidation concerning the significance of the omission of the Axiom of Choice lies outside the scope of this paper. So, we shall turn directly to general philosophical aspects concerning the mentioned consequences of the relativization of cardinalities.

5 Concluding logico-ontological considerations

Even for the hardest Platonists, the realm of higher order infinities transcending $2^{\aleph_0}$, however interesting it may be for mathematicians, seems unsurveyable from an ontological point of view. So, by dealing with philosophically interesting consequences of the Löwenheim-Skolem Theorem and Skolem's Paradox in particular, we shall focus our attention to discrete, dense and continuous structures that are sufficiently close to reality in a common sense of the word, but which involve, at the same time, the relevant difference between the countable and the uncountable.

5.1 Cardinality as a non-absolute property

As we have just seen (in 4.2), the resolution of Skolem's Paradox that has had the most important impact in mathematics demands the relativization of cardinalities. This relativization seems to be in a blatant contradiction with the very concept of cardinal number as it was originally defined by Cantor (see 1.4). Though the status of the Continuum Hypothesis allows us to take that $2^{\aleph_0} = \aleph_1$ but also that it is not so (some mathematicians have suggested that we should rather take that $2^{\aleph_0} = \aleph_2$ [37]), this does not mean that we may assume both to be the case at the same time. So, we need some reconceptualization of cardinality if it should be allowed to be non-absolute.

Now, the above consideration and the cited examples suggest a clear way in which the concept of cardinality is to be re-defined. Instead of speaking of the cardinality of a set as such, we should rather speak of the cardinality in a qualified sense, namely, of the cardinality of the basic set of a structure. In the literature, it is quite common to speak of the countability or uncountability of a model. This is not correct strictly speaking, but it can be accepted as a *façon de parler*. It is yet a set which is countable or uncountable, but it is always a basic set of a structure, which also contains relations, and the countability or the uncountability of such a set depends essentially on relations defined on it. So, instead of simply saying that a set is countable, we should always say that it is countable in view of this or that relation. Then, it becomes consistent to say that a set is uncountable in view of this but countable in view of that relation. Even the simplest example shows what this means. The set of rationals is not countable in view of the way in which the rationals are ordered by the standard precedence relation. But if we order them in a different way, by using one of the well-known functions, their set becomes countable. However trivial this may seem, it ceases to be trivial when we turn to real numbers, where under certain conditions they can be mapped onto a structure in such a way that they start to be countable (see 4.2). So, as suggested, it is not the set as such which is countable or countable but the set structured in a certain way.

5.2 Changing cardinalities: relation-dependence without re-structuring the structures

There is one thing that can be said to remain ambiguous in the just given explanation of the relativity of cardinality. It is said that the cardinality of a set can change depending on different relations that can be defined on it as the basic set. Can these different relations be assumed to hold simultaneously in a relational structure? The question is very important, for if the answer were negative, one could say that, in fact, one and the same basic set as the set of a relational structure cannot *be* uncountable and countable *at the same time*. We should say instead that, if uncountable, it can only be mapped onto a structure which is countable.

The question is tricky. On the one hand, we want to say that the set of rational numbers and (under certain conditions) the set of real numbers are countable. On the other hand, if we admit that they become countable only after appropriate re-structuring of their elements, one could say that after such a re-structuring the elements cease to be rationals or reals and become natural numbers.

I do not see any other way out but to distinguish between two senses of countability, direct and derivative, and say that the set of rational numbers and the set of real numbers are countable because there is a different structure whose basic set is directly countable and, at the same time, such that its elements can serve as images of the elements of the basic set of the original structures. After all, it is a function that maps all the rationals or reals onto a set of its images that gives the meaning to the statement that the set of rationals and reals are countable. We mustn't detach the meaning of countability from the existence of such a function, because it is actually only the set of images that is directly countable. This seems to be the only way in which we can continue to speak of rationals as rationals and of reals as reals, and to say, at the same time, that their sets are countable.

By generalizing the given example, we can say that the relativity of cardinality concerns the change of cardinality of the basic set of a structure in view of its possible mapping onto the basic set of a different structure. So, the set of real numbers does not cease to be uncountable in spite of the fact that there is a model in which the set of their images is denumerable.

5.3 *The world and its substance: relation externalism and the problem of referring to objects by re-structuring the structures*

According to Wittgenstein's Tractatus, the substance of the world is just the set of its objects [40, 2.021], whereas the world itself is a complex relational structure involving all the actual relations between objects [40, 2.022].

Now, I suggested above (in 5.2) that rational or real numbers would cease to be what they are if they were structured as natural numbers and that this represents the reason for using their images when speaking of their countability (in a derivative sense). This seems to be in a direct contradiction to Russell's relation externalism (see 1.1), according to which the objects remain what they are after having changed their relation to other objects. I think that relation externalism fails in this case only because numbers are not entities such as ordinary objects in space and time.

Let us speak, however, of the so-called rational points of a line segment, so that 0 refers to the left end-point, 1 to the right end-point, $\frac{1}{2}$ to the mid-point of the segment, and so on. Now, if we accept, at least *arguendo*, that points are basic elements of the real world, as Cantor did [5, pp. 275ff], we can imagine a real re-structuring of the points of the given line segment such that the set of its former rational points becomes directly countable. But in this case we only use rational numbers to pick out the objects (points) that supposedly exist in reality independently of how we refer to them, while in the case in which we speak of numbers themselves it is not so. By using the theory of meaning holism, we can say that we cannot refer to $\frac{1}{2}$ as an element of a relational structure by ignoring its position in the given relational structure. There is nothing like 'Venus' that could be used here to refer to $\frac{1}{2}$ directly (see 1.2).

The difference between the above two cases—numbers versus points—may become crucial when we try to apply the relativity of cardinality to the analysis of reality in a sense that is stronger that the sense in which we speak of reality of numbers. Namely, it seems that, in view of the possibility of an actual re-structuring of the continuum, there is a sense in which it could be possible that the cardinal number of the basic set actually becomes $\aleph_0$. And then again, starting from the elements of such a decomposed continuum as a pure "substance of the world," God could build up, in the inverse order, the world such as it actually is. The possibility of this scenario, which is based on the relativity of cardinality, shows that, however complex the world may be, for its constitution, it might be sufficient that its "substance" is of cardinality $\aleph_0$.

5.4 How to apply Hilbert's Programme in the formalization of God's re-structuring the elements of the space-world

As I have just suggested, if the process of changing cardinalities could have been understood as the process of a real downward decomposition of the world, it should be also possible to suppose that God, in an inverse process, has structured the real space-world by starting from a set of its basic elements whose cardinality is $\aleph_0$. The natural question is, then, how we are to proceed when trying to formalize each of the higher-order structures obtained in the process of God's re-structuring the world.

If we restrict our attention up to the stage at which one-dimensional continua have been created, the question will be reduced to the formal distinction between discrete, dense and continuous structures. However poor this might seem in view of other structures envisaged in the transfinite mathematics, it will be enough for understanding how the application of Hilbert's Programme would generally look like.

The first important question concerns language. As we saw above (see 4.1), the standard first-order language would not do the job. This means that we have to choose between some of stronger languages. For several reasons, I suggest that we use the language $L_{\omega_1\omega_1}$ [3]. One of these reasons is that we would not have to add anything to our basic assumption that the "world substance" is a set of an infinite number of elements whose cardinal number is $\aleph_0$ and would also not have to refer directly to any set at all. So, the variables will range, during the whole process of the world construction, only over the elements of one and the same basic set. The second important reason is that we shall be able to treat any of the theories formalizing a higher-order structure as a direct axiomatic extension of the lower-order theory, which will explain the relation between the non-categoricalness and incompleteness of a theory in an interesting way. This should be one of the most interesting results concerning the question we are dealing with.

The second important question concerns the Hilbertian idea that the difference between the structures we are dealing with should be a consequence of the difference in the meaning of the ordering relation, which is to be grasped only axiomatically (see 1.5). This means that, contrary to Cantor, we do not have to add anything else concerning the cardinality of the basic set as such if we find that a model of the theory formalizing a higher-order structure is non-denumerable, because (as stated in 5.1) the cardinality of a structure does not concern the basic set as such but its cardinality in view of a certain relation.

Now, if we want to formalize the linear structure that is only dense, we shall naturally add the density axiom to the rest of axioms defining implicitly a linearly ordered structure (see 4.1). The problem is, however, that the obtained formal theory has non-isomorphic models, as we can see by anticipating the next step of God's re-structuring the world. Moreover, the models will differ just in view of their cardinality!

The standard model of a dense structure is the set of rational points, whose cardinal number is $\aleph_0$. However, let us start with the unit continuum $[0, 1]$ (supposedly already created by God) and delete, in addition to its two end-points, all the open intervals $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}), (\frac{1}{27}, \frac{2}{27}), (\frac{7}{27}, \frac{8}{27}), (\frac{19}{27}, \frac{20}{27}), (\frac{25}{27}, \frac{26}{27}), \ldots$ and so on analogously. Now, since $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots + \frac{2^{n-1}}{3^n} + \ldots = 1$, the length of the deleted intervals is metrically equal to 1, while the remaining points make up a discontinuum, which is a set metrically equal to zero. This set, which is known as Cantor's ternary set, is dense (i.e., perfect in Cantor's terminology), which is easy to see. But, since the cardinal number of all the deleted intervals is supposedly greater than $\aleph_0$, the cardinal number of such a discontinuum should also be greater than $\aleph_0$, which means that it represents a non-standard model of the system containing just first eight axioms cited above (in 4.1).

Ignoring metrical differences between the basic sets of the two models—the set of rational points and Cantor's discontinuum—which can be treated as something external and irrelevant for the basic isomorphism between the models of the axiom system that contains the density axiom, the question concerning the difference in cardinality remains unsolved. Given the relativity of cardinality, on the basis of which we have supposed that the cardinal number of the basic set of elements of God's construction of the world that contains the structures of higher cardinalities is not greater than $\aleph_0$, as well as the assumption that the introduction of a higher cardinality can mean nothing else but a change of the holistic meaning of the ordering relation, we must try, by pursuing the Hilbertian approach, to grasp this change axiomatically.

Contrary to standard dense but not continuous structures, Cantor's discontinuum as a specific, non-standard discontinuous structure, which contains "wholes" that are continuous, can be expressed only in a system that contains axioms (9) and (10) (see 4.1), which implicitly define structures of a higher cardinality. This means that the system containing just first eight axioms must be said to be incomplete, since it is non-trivially extendable though the introduction of new axioms. So, the specific non-categoricalness of the system defining dense structures can be overcome if we complete it in one way or another by using axioms (9) and (10) or their negations. Let us mention that we can get interesting non-standard models only one of the two axioms [3, 3.3].

But again, if we add axioms (9) and (10) that are necessary for obtaining linear continua, we obtain a formal theory that is further completable in different ways! In particular, the system containing 10 axioms cited above (in 4.1) can be extended through the introduction of large-scale and small-scale Archimedean axioms as well as through the introduction of the non-Archimedean ones. So, for instance, we can preclude the non-standard interpretation by introducing the following two axioms [3, p. 42]:

$$\exists\alpha_1\exists\alpha_2\ldots\exists\alpha_n\ldots(\alpha_2<\alpha_1\wedge\bigwedge_{1\leq i<\omega}\alpha_{2i-1}<\alpha_{2i+1}\wedge\bigwedge_{1\leq i<\omega}\alpha_{2i+2}<\alpha_{2i}\wedge$$
$$\wedge\forall\beta\bigwedge_{1\leq i<\omega}((\alpha_i<\beta\wedge\beta<\alpha_{i+2})\rightarrow\bigwedge_{1\leq k<\omega}\neg\beta=\alpha_k)\wedge$$
$$\wedge\forall\gamma\bigvee_{1\leq i,j<\omega}(\alpha_i<\gamma\wedge\gamma<\alpha_j))$$

and

$$\exists\alpha_1\ldots\exists\alpha_n\ldots$$
$$(\forall\beta\bigvee_{1\leq i,j<\omega}(\alpha_i<\beta\wedge\beta<\alpha_j)\wedge\forall\gamma\forall\delta(\gamma<\delta\rightarrow\bigvee_{1\leq i<\omega}(\gamma<\alpha_k\wedge\alpha_k<\delta)))$$

where the first of them precludes the hyper-finite elements, while the second one precludes infinitesimals.

And so on, and so forth. It is clear in which way we are to cope with the problem of non-categoricalness. Though it is not possible to settle this problem once for ever, whenever it can be reduced to the incompleteness question, we can apply the *pay-as-you-go* strategy and preclude the unintended non-isomorphic models by a concrete extension of the theory. And in a possible case in which it were not possible to proceed in this way any longer, the problem should be re-considered as the question concerning the weakness of the formal language we have used.

And finally, as for the question about cardinalities of different structures we come across by following God's re-structuring the world, it must be admitted that it cannot be answered in a straightforward way, since it is not clear how exactly mathematicians themselves use the concept of cardinality after its re-conceptualization in view of the Löwenheim-Skolem Theorem and the cited results of Cohen, Feferman and Levy. As I suggested above, only discrete structures like $\langle\mathbb{N},\leq\rangle$, should be said to be directly denumerable. But this does not mean that there are not special reasons for distinguishing various kinds of countability in the derivative sense. Mathematical reality is more complex than the reality of the Cantorian space-world we have been talking about. It would be interesting to see which reasons have led some of mathematicians to assume that $2^{\aleph_0}=\aleph_2$ [37], but such an investigation lies outside the scope of this paper.

Acknowledgements I am very much thankful to Miloš Adžić for discussions about the topic and an anonymous referee for useful comments on an earlier draft of the paper.

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