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THE 'GREAT STRUGGLE' BETWEEN CANTORIANS AND NEO-ARISTOTELIANS: MUCH ADO ABOUT NOTHING

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Summary
Starting from the generalized concept of syntactically and semantically trivial differences between two formal theories introduced by Arsenijević, we show that two systems of the linear continuum, the Cantorian point-based system and the Aristotelian interval-based system that satisfies Cantor's coherence condition, are only trivially different. So, the 'great struggle' (to use Cantor's phrase) between the two contending parties turns out to be 'much ado about nothing'.

1. *Introduction*

According to what Aristotle called *Zeno's axiom*,[1] no *n*-dimensional entity *consists solely of* (*n*-1)-dimensional entities. Accepting Zeno's axiom and rejecting atomism at the same time, Aristotle established an *interval-based* conception of the continuum that 'involves indeterminate parts' and was therefore later called 'indefinitism'.[2] This conception was the official conception of the continuum till the end of the nineteenth century, although mathematicians often used infinitesimals as 'useful fictions',[3] and physicists sometimes endorsed atomism (but not as an analysis of the continuum).

Because the competing '*theorists* [...] either leave ultimate elements of matter totally indeterminate, or [...] they assume them to be so-called *atoms* of *very small*, yet *not entirely disappearing space-contents*', the 'great struggle' among the followers of Aristotle and Epicurus so scandalized Georg Cantor that he was unwilling to let the matter go unresolved.[4] Boldly rejecting Zeno's axiom, Cantor established the *point-based* conception

1. See Aristotle, *Metaphysics* 1001 b 7.
2. See Leibniz G., II, pp. 281–83.
3. See ibid. GM, IV, p. 91–95.
4. Cantor 1962, p. 275.

of the continuum, stating that a linearly ordered set of null-dimensional points actually makes up a continuum if the set is *perfect* and *coherent* (zusammenhängend),[5] which means that each element of the set is an accumulation point of an infinite number of elements of the set, whereas each accumulation point of an infinite number of elements of the set is an element of the basic set itself.

The revival of infinitesimalism[6] and the formalization of the non-Archimedean system of the continuum[7] did not prevent Cantorians from dominating twentieth-century mathematics just as Aristotelians had dominated the subject till the end of the nineteenth century. Cantor's theory has been enormously influential. Logicians have formalized it, mathematicians have accepted it as a basis for Standard Analysis, and because of it philosophers have changed their mind about the structure of the physical world. Even physicists haven't quantized space and time, in spite of the fact that they acknowledge the existence of the quantum of action.

But though the majority of mathematicians and scientists sided with Cantor's view, and many prominent philosophers did considerable work to defend it as the ontology of the physical world,[8] in the last three decades of the last century a number of authors revived the Aristotelian stretch-based approach.[9]

However, in his 2003 article Arsenijević[10] argued that there are interesting cases in which two axiomatic systems, or two formal theories, which are syntactically and semantically non-trivially different in the standard sense of these terms should be rather classified as only trivially different. What we now want to prove is that the point-based and the interval-based system of the continuum represent a remarkable instance of such a case, so that the 'great struggle' between Cantorians and Neo-Aristotelians turns out to be 'much ado about nothing'.

Arsenijević has formulated two sets of translation rules that he has shown[11] to be sufficient for the mutual translatability of the formulas of

5. Ibid. p. 190.

6. See Ehrlich 2005.

7. Robinson, A. 1974.

8. For instance, Russell 1903 and 1914, Carnap 1928, Grünbaum 1952 and 1974, Salmon 1975, Robinson, D. 1989, Lewis 1994, Earman and Roberts 2006.

9. Hamblin 1969 and 1971, Humberstone 1979, Foldes, 1980, Needham 1981, Burgess 1982, Comer 1985, White 1988, Bochman 1990a and 1990b, Benthem van 1991 and 1995, Roeper 1997, and 2006.

10. Arsenijević 2003, pp. 2–4.

11. Ibid. pp. 8–9.

the point-based and the interval-based systems when their axioms are so selected that they implicitly define linearly ordered dense structures. The leading idea of Arsenijević's proof that these systems are trivially different consists in using as the basis of the translations the so-called Felix Bernstein's mapping[12] between the two sets of formulas, which is a 1-1 mapping of all the formulas of each of the two sets into (and not onto) the set of formulas of the other set.[13]

From a purely syntactical point of view, it is sufficient that, in both directions, each theorem, but no non-theorem, is translated by a theorem. From a semantic point of view, however, it is not sufficient that, in both directions, each truth, but no falsehood, is translated by a truth, since it is necessary that each translation also be structure preserving. This means that both the elements and relations of a model of one of the systems must be unequivocally spoken of in terms of the elements and relations of a model of the other system. In the case at hand, the foregoing requirement will be satisfied because any element of the interval structure will be unequivocally identified and spoken of as a *stretch* between two distinct points of the point structure, whereas each element of the point structure will be unequivocally identified and spoken of as an *abutment place* of two abutting stretches of the stretch structure. The identity and precedence relations of either of the two structures will be unequivocally definable *via* the identity and precedence relations of the other structure in the way in which it is done below (see translation rules C_1 and C_2, and also C^*_1 and C^*_2).

Now, the main problem of using the translation rules formulated by Arsenijević lies in the fact that those rules are tailored to first-order languages, whereas the continuity axiom (defining implicitly the coherence of the set of points and the set of stretches) is normally formulated in a second-order language. In order to avoid this problem, we shall use the $L\omega_1\omega_1$ language to express the continuity axiom in both systems. These formulations will make it possible to extend the applicability of the translation rules formulated for the first-order languages without any modification.

As for the difference between translation rules concerning quantifiers (C_5 and C^*_5) as originally formulated by Arsenijević and as they are formulated below,[14] the new formulation has an obvious advantage in simplicity,

12. See Cantor 1962, p. 450.

13. See Arsenijević 2003, pp. 3, 8–9.

14. Compare the two original formulations in Arsenijević 2003, pp. 8–9 with the two given below.

but it will also enable us to translate the continuity axioms of the two systems into a considerably shorter form (see the comments below about the shorter form of translating the closed formulas).

In the next Section, we shall desribe the two formal theories and cite the re-formulated translation rules in full, so that the article will be self-contained.

2. *Comparison between the point-based and the interval-based systems of the continuum according to the generalized definition of the trivial differences between formal theories*

Let S_p contain—besides the logical constants $\neg$, $\Rightarrow$, $\wedge$, $\vee$ and $\Leftrightarrow$—individual variables $\alpha_1, \alpha_2,\ldots, \alpha_i,\ldots, \beta_1, \beta_2,\ldots, \beta_i,\ldots, \gamma_1, \gamma_2,\ldots, \gamma_i,\ldots, \delta_1, \delta_2,\ldots, \delta_i,\ldots$, quantifiable by universal and existential quantifiers. The variables are supposed to range over a set of null-dimensional points. Also let S_P contain relation symbols $\equiv$, $<$ and $>$, to be interpreted as the identity, precedence, and succession relations respectively. Let the elementary *wffs* of S_P be $\alpha_m \equiv \alpha_n$, $\alpha_m < \alpha_n$ and $\alpha_m > \alpha_n$, where $a_m > a_n \Leftrightarrow_{\text{def.}} a_n < a_m$. Finally, let axiom schemes of S_P be the following ten formulas, which we shall refer to as (A_P1), (A_P2),…, (A_P10):

1. $(\alpha_n)\neg\alpha_n < \alpha_n$
2. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l < \alpha_m \wedge \alpha_m < \alpha_n \Rightarrow \alpha_l < \alpha_n)$
3. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \vee \alpha_n < \alpha_m \vee \alpha_m \equiv \alpha_n)$
4. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_l < \alpha_n \Rightarrow \alpha_m < \alpha_n)$
5. $(\alpha_l)(\alpha_m)(\alpha_n)(\alpha_l \equiv \alpha_m \wedge \alpha_n < \alpha_l \Rightarrow \alpha_n < \alpha_m)$
6. $(\alpha_m)(\exists\alpha_n)\alpha_m < \alpha_n$
7. $(\alpha_m)(\exists\alpha_n)\alpha_n < \alpha_m$
8. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \Rightarrow (\exists\alpha_l)(\alpha_m < \alpha_l \wedge \alpha_l < \alpha_n))$
9. $(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots\ ((\exists\beta_1)(\wedge_{1\leq i<\omega}\ \alpha_i < \beta_1) \Rightarrow$
   $\Rightarrow (\exists\gamma_1)(\wedge_{1\leq i<\omega}\ \alpha_i < \gamma_1 \wedge\neg(\exists\delta_1)(\wedge_{1\leq i<\omega}\ \alpha_i < \delta_1 \wedge \delta_1 < \gamma_1)))$
10. $(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots\ ((\exists\beta_1)(\wedge_{1\leq i<\omega}\ \alpha_i > \beta_1) \Rightarrow$
    $\Rightarrow (\exists\gamma_1)(\wedge_{1\leq i<\omega}\ \alpha_i > \gamma_1 \wedge \neg(\exists\delta_1)(\wedge_{1\leq i<\omega}\ \alpha_i > \delta_1 \wedge \delta_1 > \gamma_1)))$

Let S_I contain—besides the logical constants $\neg$, $\Rightarrow$, $\wedge$, $\vee$ and $\Leftrightarrow$—individual variables $a_1, a_2,\ldots, a_i,\ldots, b_1, b_2,\ldots, b_i,\ldots, c_1, c_2,\ldots, c_i,\ldots, d_1, d_2,\ldots, d_i,\ldots$, quantifiable by universal and existential quantifiers. The variables are supposed to range over one-dimensional stretches. Also let S_P contain the relation symbols $=$, $\prec$, $\succ$, $\wr$, $\cap$ and ☪, to be interpreted as the identity, (total) precedence, (total) succession, abutment, overlapping and inclusion relations respectively. Then, the elementary *wffs* will be $a_m = a_n$, $a_m \prec a_n$, $a_m \succ a_n$, $a_m \wr a_n$, $a_m \cap a_n$, and $a_m \text{ ☪ } a_n$. But though it will be sometimes useful and much easier to use all these relations, it is important to note that not only $\succ$, but also $\wr$, $\cap$ and ☪, are definable *via* $=$ and $\prec$ in the following way:

$$a_m \succ a_n \Leftrightarrow_{\text{def.}} a_n \prec a_m,$$
$$a_m \wr a_n \Leftrightarrow_{\text{def.}} a_m \prec a_n \wedge \neg(\exists a_l)(a_m \prec a_l \wedge a_l \prec a_n),$$
$$a_m \cap a_n \Leftrightarrow_{\text{def.}} (\exists a_l)(\exists a_k)(a_l \prec a_n \wedge \neg a_l \prec a_m \wedge a_m \prec a_k \wedge \neg a_n \prec a_k),$$
$$a_m \text{ ☪ } a_n \Leftrightarrow_{\text{def.}} \neg a_m = a_n \wedge (a_l)(a_l \cap a_m \Rightarrow a_l \cap a_n).$$

Finally, let axiom schemes of S_I be the following ten formulas, which we shall refer to as (A_I1), (A_I2),…, (A_I10):

1. $(a_n)\neg a_n \prec a_n$
2. $(a_k)(a_l)(a_m)(a_n)(a_k \prec a_m \wedge a_l \prec a_n \Rightarrow a_k \prec a_n \vee a_l \prec a_m)$
3. $(a_m)(a_n)(a_m \prec a_n \Rightarrow a_m \wr a_n \vee (\exists a_l)(a_m \wr a_l \wedge a_l \wr a_n))$
4. $(a_k)(a_l)(a_m)(a_n)(a_k \wr a_m \wedge a_k \wr a_n \wedge a_l \wr a_m \Rightarrow a_l \wr a_n)$
5. $(a_k)(a_l)(a_m)(a_n)(a_k \wr a_l \wedge a_l \wr a_n \wedge a_k \wr a_m \wedge a_m \wr a_n \Rightarrow a_l = a_m)$
6. $(a_m)(\exists a_n)\ a_m \prec a_n$
7. $(a_m)(\exists a_n)\ a_n \prec a_m$
8. $(a_m)(\exists a_n)\ a_n \text{ ☪ } a_m$
9. $(a_1)(a_2)\ldots(a_i)\ldots((\exists u)(\wedge_{1 \leq i < \omega}\ a_i \prec u) \Rightarrow$
   $\Rightarrow (\exists v)\ (\wedge_{1 \leq i < \omega}\ a_i \prec v \wedge \neg(\exists w)\ (\wedge_{1 \leq i < \omega}\ a_i \prec w\ \wedge w \prec v)))$
10. $(a_1)(_2)\ldots(a_i)\ldots((\exists u)(\wedge_{1 \leq i < \omega}\ a_i \succ u) \Rightarrow$
    $\Rightarrow (\exists v)(\wedge_{1 \leq i < \omega} a_i \succ v \wedge \neg(\exists w)(\wedge_{1 \leq i < \omega} a_i \succ w\ \wedge w \succ v)))$

Now, let f be a function

$$f\colon \alpha_n \longrightarrow \langle a_{2n-1}, a_{2n}\rangle \ (n = 1, 2, \ldots)$$

mapping variables of S_P into ordered pairs of variables of S_P, and let C_1–C_5 be the following translation rules providing a 1-1 translation of all the *wffs* of S_P into a subset of the *wffs* of S_I (where $=^C$ means "is to be translated according to syntactic constraints C as"):

C_1:

$\alpha_n \equiv \alpha_m =^C a_{2n-1} \wr a_{2n} \wedge a_{2m-1} \wr a_{2m} \wedge a_{2n-1} \wr a_{2m}$,

C_2:

$\alpha_n < \alpha_m =^C a_{2n-1} \wr a_{2n} \wedge a_{2m-1} \wr a_{2m} \wedge a_{2n-1} \prec a_{2m} \wedge \neg a_{2n-1} \wr a_{2m}$,

C_3:

$\neg F_P =^C \neg C(F_P)$, where F_P is a *wff* of S_P translated according to C_1–C_5 into *wff* $C(F_P)$ of S_I,

C_4:

$F_P' \heartsuit F_P'' =^C C(F_P') \heartsuit C(F_P'')$, where ♥ stands for $\Rightarrow$ or $\wedge$ or $\vee$ or $\Leftrightarrow$, and F_P' and F_P" stand for two *wffs* of S_P translated according to C_1–C_5 into two *wffs* of S_I, $C(F_P')$ and $C(F_P'')$ respectively,

C_5:

$(\alpha_n)\Omega(\alpha_n) =^C (a_{2n-1})(a_{2n})((a_{2n-1} \wr a_{2n}) \Rightarrow \Omega^*(a_{2n-1}, a_{2n}))$ and
$(\exists\alpha_n)\Omega(\alpha_n) =^C (\exists a_{2n-1})(\exists a_{2n})((a_{2n-1} \wr a_{2n}) \wedge \Omega^*(a_{2n-1}, a_{2n}))$,
where $\Omega(\alpha_n)$ is a formula of S_P translated into formula $\Omega^*(a_{2n-1}, a_{2n})$ of S_I according to C_1–C_5 .

Let f^* be a function

$$f^* : a_n \longrightarrow \langle \alpha_{2n-1}, \alpha_{2n} \rangle \ (n = 1, 2, \ldots)$$

mapping variables of S_I into ordered pairs of variables of S_P, and let C^*_1–C^*_5 be the following translation rules providing a 1-1 translation of all the *wffs* of S_I into a subset of the *wffs* of S_P (where $=^{C^*}$ is to be understood analogously to $=^C$):

C^*_1:

$a_n = a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \alpha_{2n-1} \equiv \alpha_{2m-1} \wedge \alpha_{2n} \equiv \alpha_{2m}$,

C^*_2:

$a_n \prec a_m =^{C^*} \alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \neg\alpha_{2m-1} < \alpha_{2n}$,

C^*_3:

$\neg F_I =^{C^*} \neg C^*(F_I)$, where F_I is a *wff* of S_I translated according to C^*_1–C^*_5 into *wff* $C(F_I)$ of S_P,

C^*_4:

$F_I' \heartsuit F_I'' =^{C^*} C^*(F_I') \heartsuit C^*(F_I'')$, where ♥ stands for $\Rightarrow$ or $\wedge$ or $\vee$ or $\Leftrightarrow$, and F_I' and F_I" stand for two *wffs* of S_I translated according to C^*_1–C^*_5 into two *wffs* of S_P, $C^*(F_I')$ and $C^*(F_I'')$ respectively,

C^*_5:

$(a_n)\Phi(a_n) =^{C^*} (\alpha_{2n-1})(\alpha_{2n})((\alpha_{2n-1} < \alpha_{2n}) \Rightarrow \Phi^*(\alpha_{2n-1}, \alpha_{2n}))$ and

$(\exists a_n)\Phi(a_n) =^{C^*} (\exists\alpha_{2n-1})(\exists\alpha_{2n})((\alpha_{2n-1} < \alpha_{2n}) \wedge \Phi^*(\alpha_{2n-1}, \alpha_{2n}))$,
where $\Phi(a_n)$ is a formula of S_I translated into formula
$\Phi^*(\alpha_{2n-1}, \alpha_{2n})$ of S_P according to C^*_1–C^*_5.

These translation rules (C_1–C_5 and C^*_1–C^*_5) constitute an effective mechanical procedure for translating any formula of either of the two systems, be it open or closed, into exactly one formula of the other system. However, since by translating closed formulas the condition occurring in the translation of quantifiers reoccurs necessarily either as a conjunct or as a part of the consequent of the translation (depending on whether the existential or the universal quantifier is involved in a given translation), we can always obtain an equivalent but shorter version of the resulting formula. Since it is obvious why it is so if the quantifier of the original formula is existential, let us take an example when it is universal. Let $\Phi(a_n)$ be (A_I1), that is

$$(a_n)\neg(a_n \prec a_n).$$

According to C^*_1–C^*_5, the translation $\Phi^*(\alpha_{2n-1}, \alpha_{2n})$ of (A_I1), to be denoted as (A_I1)*, reads as follows:

$$(\alpha_{2n-1})(\alpha_{2n})(\alpha_{2n-1} < \alpha_{2n} \Rightarrow \neg(\alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2n-1} < \alpha_{2n} \wedge \neg\alpha_{2n-1} < \alpha_{2n})).$$

However, this is equivalent to

$$(\alpha_{2n-1})(\alpha_{2n})(\alpha_{2n-1} < \alpha_{2n} \Rightarrow \neg\neg\alpha_{2n-1} < \alpha_{2n}),$$

since $\alpha_{2n-1} < \alpha_{2n} \wedge \alpha_{2n-1} < \alpha_{2n}$ occurring in the consequent can be erased, given that in propositional calculus $p \Rightarrow \neg(p \wedge p \wedge q)$ is equivalent to $p \Rightarrow \neg\, q$.

Now, by using this device for obtaining the shorter form of a translation, we get (A_P9)* as a translation of the axiom (A_P9):

(A_P9)*
$$\begin{aligned}&(a_1)(a_2)\ldots(a_i)\ldots\ (\wedge_{1\leq i<\omega}\, a_{2i-1} \{\, a_{2i} \Rightarrow ((\exists b_1)(\exists b_2)(\, b_1 \{\, b_2 \wedge \\ &\quad \wedge (\wedge_{1\leq i<\omega}\, a_i \prec b_2)) \Rightarrow \\ &\Rightarrow (\exists c_1)(\exists c_2)(c_1 \{\, c_2 \wedge (\wedge_{1\leq i<\omega}\, a_i \prec c_2) \wedge \\ &\wedge\neg(\exists d_1)\neg(\exists d_2)(d_1 \{\, d_2 \wedge ((\wedge_{1\leq i<\omega}\, a_i \prec d_2) \wedge d_1 \prec c_2 \wedge \neg\, d_1 \{ c_2)))).\end{aligned}$$

In the same way, we get $(A_I9)^*$ as a translation of the axiom (A_I9):

$(A_I9)^*$

$$(\alpha_1)(\alpha_2)\ldots(\alpha_i)\ldots\ (\wedge_{1\leq i<\omega}\ \alpha_{2i-1} < \alpha_{2i} \Rightarrow ((\exists\beta_1)(\exists\beta_2)(\ \beta_1 < \beta_2 \wedge$$
$$\wedge\ (\wedge_{1\leq i<\omega} \neg\beta_1 < \alpha_{2i})) \Rightarrow$$
$$\Rightarrow (\exists\gamma_1)(\exists\gamma_2)(\gamma_1 < \gamma_2 \wedge (\wedge_{1\leq i<\omega} \neg\gamma_1 < \alpha_{2i}) \wedge$$
$$\wedge\neg(\exists\delta_1)\neg(\exists\delta_2)(\delta_1 < \delta_2 \wedge ((\wedge_{1\leq i<\omega} \neg\delta_1 < \alpha_{2i}) \wedge \neg\gamma_1 < \delta_2)))).$$

The main thing to do is to prove that both $(A_P9)^*$ and $(A_P10)^*$, as well as $(A_I9)^*$ and $(A_I10)^*$, are theorems of S_I and S_P, respectively.

Proof for $(A_P9)^*$
Let us assume both

$$\wedge_{1\leq i<\omega}\ a_{2i-1} \wr a_{2i} \quad \text{and} \quad (\exists b_1)(\exists b_2)(\ b_1 \wr b_2 \wedge (\wedge_{1\leq i<\omega}\ a_i \prec b_2)),$$

which are the two antecedents of $(A_P9)^*$. Now, since for any i $(1\leq i<\omega)$, $a_i \prec b_2$, it follows directly from (A_I9) that there is v such that $a_i \prec v$ and, for no w, both $a_i \prec w$ and $w \prec v$.

Let us now assume, contrary to the statement of the consequent of $(A_P9)^*$, that for any two c_1, c_2 such that $c_1 \wr c_2$ and for any i $(1\leq i<\omega)$ $a_i \prec c_2$, there are always d_1 and d_2 such that $d_1 \wr d_2$ and for any i $(1\leq i<\omega)$ $a_i \prec d_2$, so that $d_1 \prec c_2$ and $\neg\ d_1 \wr c_2$. But then, if we take c_2 to be just v from the consequent of (A_I9) (and c_1 any interval such that $c_1 \wr c_2$), the assumption that for any i $(1\leq i<\omega)$ $a_i \prec c_2$ but $d_1 \prec c_2$ and $\neg\ d_1 \wr c_2$ contradicts the choice of c_2, since if $c_2 = v$, then, according to (A_I9), for any d_1 and d_2 such that $d_1 \wr d_2$ and for any i $(1\leq i<\omega)$ $a_i \prec d_2$, it cannot be that $d_1 \prec c_2$ and $\neg\ d_1 \wr c_2$. (Q.E.D.)

Proof for $(A_I9)^*$
Let us assume both

$$\wedge_{1\leq i<\omega}\ \alpha_{2i-1} < \alpha_{2i} \quad \text{and} \quad (\exists\beta_1)(\exists\beta_2)(\ \beta_1 < \beta_2 \wedge (\wedge_{1\leq i<\omega} \neg\beta_1 < \alpha_{2i})),$$

which are the two antecedents of $(A_I9)^*$. Now, since for any i $(1\leq i<\omega)$, $\neg\beta_1 < \alpha_{2i}$ implies $\neg\beta_1 < \alpha_i$, it follows directly from (A_P9) that there is γ such that $\neg\gamma < \alpha_i$ and for no δ, both $\neg\delta < \alpha_i$ and $\delta < \gamma$.

Let us now assume, contrary to the statement of the consequent of $(A_I9)^*$, that for any two γ_1, γ_2 such that $\gamma_1 < \gamma_2$ and for any i $(1\leq i<\omega)$ $\neg\gamma_1 < \alpha_{2i}$, there are always δ_1 and δ_2 such that $\delta_1 < \delta_2$ and for any i $(1 \leq i < \omega)$ $\neg\delta_1 < \alpha_{2i}$, so that $\neg\delta_1 < \alpha_{2i}$ and $\neg\gamma_1 < \delta_2$. But then, if we take γ_1 to be just γ from the consequent of (A_P9) (and γ_2 any point such that $\gamma_1 < \gamma_2$), we get first $\neg\gamma_1 < \delta_2$, and then $\delta_1 < \gamma_1$ (since $\delta_1 < \delta_2$), which contradicts the choice of γ_1, since if $\gamma_1 \equiv \gamma$, there is no δ (and so also no δ_1) such that both $\neg\delta < \alpha_i$ and $\delta < \neg\gamma_1$. (Q.E.D.)

It can proved analogously that $(A_P10)^*$ and $(A_I10)^*$ are also theorems of S_I and S_P, respectively.

3. *Conclusion*

Because of the intuitive similarity between the point-based and the interval-based systems of the continuum, it is hard to believe, in spite of the 'great struggle' between Cantorians and Aristotelians, that nobody else had the idea that the two systems of the continuum are not so radically different that there wouldn't be some sense in which one could say that they are merely trivially different. And yes, by speaking about instant-based and period-based time systems, van Benthem has proclaimed that 'systematic connections between point structures and period structures enable one to use both perspectives at will'.[15] But, van Benthem says nothing concrete about how these 'systematic connections' are to be formally defined and how it is to be proved that the point-based and the interval-based systems conform to such a definition.

Now, in Section 2 it is established that after translating (A_P9) into S_I and (A_I9) into S_P the obtained formulas $(A_P9)^*$ and $(A_I9)^*$ are theorems of S_I and S_P respectively. The same holds for $(A_P10)^*$ and $(A_I10)^*$. Together with Arsenijević's analogous 2003 result concerning the first eight axioms of the two systems, this is sufficient for suggesting that S_P and S_I are *syntactically* only trivially different. However, more needs to be said about the alleged *semantically* trivial difference between them, for there is a seemingly striking discrepancy between the entities of their corresponding models.

The basic elements of the intended model of S_P are points, whereas intervals are continuous sets of points. The basic elements of the intended model of S_I are stretches, whilst points are the abutment places of abutting stretches. But aren't *stretches* (of S_I) and *intervals* (of S_P) hope-

15. Van Benthem 1991, p. 84.

lessly different entities, given that stretches are *neither* open *nor* closed *nor* half-open (half-closed), whereas intervals are necessarily *either* open *or* closed *or* half-open (half-closed)? The solution to this problem can be found in the way in which in (A_I9) and (A_I10) the suppositions for the existence of the least upper bound and the greatest lower bound are introduced. Namely, an infinite set of stretches *having the least upper bound* correspond to an *interval open on the right side* in the point-based structure, whereas, analogously, an infinite number of stretches *having the greatest lower bound* correspond to an *interval open on the left side*. Consequently, an infinite number of stretches *having both the least upper and the greatest lower bound* correspond to an *open interval*. And then, curiously enough, *stretches themselves*, which are *originally* neither closed nor open, *turn out* to correspond to *closed intervals* in the point-based structure.

There is another big philosophical question that has to be answered. According to Quine's famous slogan 'To be assumed as an entity is to be reckoned as the value of a variable'[16], the two formal theories are *not trivially* different if there is no model in which their variables range over the elements of one and the same basic set, and in the case of S_P and S_I their variables can never do this. But why should it be so important what *variables do*, if the *set of entities*—elementary and non-elementary—is the *same* in *any intended model* of the two theories?

So, *pace* Quine, there are good reasons for saying that S_P and S_I are only trivially different. Syntactically, it is sufficient that there are two sets of translation rules that, though not inverses of one another, are theorem preserving. Semantically, it is sufficient that when speaking about points we cannot avoid *automatically saying something unequivocal* about stretches, and *vice versa*.[17]

Finally, let us mention a benign asymmetry between the two systems. When we speak of a least upper bound in a continuous point-based structure, it is always just a single point. However, the least upper bound in a continuous interval structure is not a single interval but an equivalence class of intervals. This trivial fact is a consequence of another trivial fact. Contrary to a stretch, which is always a single interval of an interval structure, the place of the abutment of any two intervals, which defines a point of a point structure, is always an abutment place of an infinite number of

16. Quine 1961, p. 13.
17. See Arsenijević 2003, pp. 10–11.

intervals. But we can always choose one pair from the equivalence class of abutting intervals to represent a given point.[18]

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