SOLUTION OF THE STACCATO VERSION OF THE ACHILLES PARADOX

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§1. PROBLEM

Let $\underline{P}_1$ and $\underline{P}_2$ be the following premises of a syllogism:

($\underline{P}_1$) After being in rest-state $\mathcal{A}_0$ (touching at A the area he is to move through), Achilles has moved along path AB according to the following pattern: a 1/2 h movement which covers one half of AB, followed by a 1/2 h rest (in rest-state $\mathcal{A}_1$, touching at the mid-point of AB the area in front of him); a 1/4 h movement which covers the distance between 1/2 AB and 3/4 AB, followed by a 1/4 h rest (in rest-state $\mathcal{A}_2$), and so on.., a $1/2^{n+1}$ h movement which covers the distance between $(2^n-1)/2^n$ AB and $(2^{n+1}-1)/2^{n+1}$ AB, followed by a $1/2^{n+1}$ h rest;... (n=0, 1, 2,...).

($\underline{P}_2$) 2 h after the beginning of his <u>staccato</u> motion (described in $\underline{P}_1$), Achilles is definitely at rest.

If $\underline{P}_1$ and $\underline{P}_2$ are true, what are we to conclude about the position of Achilles and the number of his rest-states after the 2 h have elapsed?

It seems obvious that Achilles cannot be anywhere between A and B, because for any such position we can demonstrate that Achilles has had to leave it before the 2 h have elapsed. It can be also shown that he could have reached no further than B.

A. Pavković (ed.), Contemporary Yugoslav Philosophy: The Analytic Approach, 27–55.

Concerning the number of rest-states, no finite number seems to be an adequate answer: for any k, however great, there is a point $(2^{k+1}-1)/2^{k+1}$ at which Achilles has had to be in rest-state $\mathcal{A}_{k+1}$.

So it seems that if Achilles has moved from A towards B in accordance with P_1, and if P_2 is true, we have to conclude that

(C) Achilles has reached B, after an infinite number of stops.

However, even though it seems that C is the conclusion which follows from premises P_1 and P_2 that can be true, unsurmountable difficulties are encountered with $\diamond C$.

For C to become true, Achilles has to perform an infinite number of strictly discrete successive acts of non-infinitesimal duration in a limited period of time. However,

a) the finiteness of the number of performed acts – continuous motions – is recursively preserved (where the fact that both the time interval for the performance of these acts and the duration of the rests decrease and converge to zero does not change anything). How could Achilles free himself from this property of the finiteness of the number of performed acts, and 'jump' into the situation of having performed an infinite number of them?

b) There is neither enough time within the 2 h period nor enough room within AB for an infinite number of movements defined in P_1: for every possible instant within the 2 h period (and every possible point within AB) it holds that up to that instant (and that point) Achilles has performed a finite number of movements. How could one single rest-state – the state of Achilles' being at B – change the situation so radically that it becomes true that Achilles has performed an infinite number of movements?

c) The geometric progression whose general member is

$$[(2^n-1)/2^n,\ (2^{n+1}-1)/2^{n+1}]$$

has no last member. How could Achilles find himself at B after having performed the series of strictly discrete acts of non-infinitesimal duration without having performed a last act?

This profound problem is typical of philosophy: the inevitable conclusion which seems untenable is drawn from the premises which seem as statements which evidently can be true.

If the problem arises from the fact that P_1 and P_2 seem possibly true – $\diamond P_1$ and $\diamond P_2$ – while at the same time conclusion C, which follows from P_1 and

$\underline{P}_2$ - $\underline{P}_1$, $\underline{P}_2$ $\vdash \underline{C}$ - seems necessarily false - $\neg\Diamond\underline{C}$ - then it seems that the only way out is to deny somehow $\Diamond\underline{P}_1$, or $\Diamond\underline{P}_2$, or $\Diamond\underline{P}_1$, $\Diamond\underline{P}_2$ $\vdash \Diamond\underline{C}$, or $\neg\Diamond\underline{C}$. The disjunction need not be exclusive.

We shall examine in detail the known proposals for the solution of The Achilles, bearing in mind that in some cases their authors did not tackle the problem in the form in which we have presented it, but were confronted with some related versions of it. The last fact is of importance in those cases where we do not obtain the answer to the problem presented in our version of it despite elaborate solutions of some other versions of it.

§2. PROPOSALS

The known proposals for the solution of The Achilles can be divided into four groups:

I. those which deny both $\Diamond\underline{P}_1$ and $\Diamond\underline{P}_2$;

II. those which deny $\Diamond\underline{P}_1$;

III. those which accept $\Diamond\underline{C}$;

IV. those which, at the same time, accept, or have no way to avoid, $\Diamond\underline{P}_1$ and $\Diamond\underline{P}_2$, and deny $\Diamond\underline{C}$, and yet do not show what may be wrong with $\Diamond\underline{P}_1$, $\Diamond\underline{P}_2$ $\vdash\underline{C}$.

I. $\boxed{\neg\Diamond\underline{P}_1 \ \& \ \neg\Diamond\underline{P}_2}$

Negative Dialectic

In Plato's Parmenides, Socrates concludes - without Zeno's explicit agreement (Zeno being present in the discussion) that he, Socrates, is right at that point - that all of Zeno's arguments against plurality are aimed at the confirmation of Parmenides' monistic ontology[1] .

If we accept Socrates' reasoning, we can say that The Achilles is aimed at the confirmation of the non-existence of motion, and if we consider, in particular, our version of The Achilles, the rationale could be the following: if we accept that Achilles can move, we cannot deny that $\underline{P}_1$ is possible, and by adding $\underline{P}_2$, which seems evidently possible if $\underline{P}_1$ is possible, we get the paradoxical $\underline{C}$. Thus, in order to avoid the paradoxical consequence, we have to conclude that motion is impossible, with $\neg\Diamond\underline{P}_1$. In addition, in the world in which motion is impossible, time does not exist[2], thus also $\neg\Diamond \underline{P}_2$.

If we admit that if motion is at all possible there is no way to disallow the possibility of $\underline{P}_1$ and $\underline{P}_2$ to be true, and if, in addition, $\underline{C}$ is paradoxical, The Achilles, much as Zeno's dialectical proofs in general, results either in sceptical resignation or in the negation of the possibility of the existence of the world of change. This last result, expressed radically in Gorgias' teaching on ontological nihilism[3], explains the origin of label negative dialectic[4].

The idealist F.H. Bradley claimed that the acceptance of the non-existence of the world of change does not imply that nothing exists. Moreover, he asserted that the selfcontradictory world of change can be the appearance of the timeless Absolute[5].

Bradley does not explain 'how the appearances (in time and space) come to be, and again how without contradiction they can be real in the Absolute', but he insists 'that such knowledge is not necessary', because 'what we require to know is only that these appearances are not incompatible with our Absolute...Since it is possible that these appearances can be resolved into a harmony which both contains and transcends them; since again it is necessary, on our main principle, that this should be so – it therefore truly is real'[6].

Criticising this conclusion, Alfred Ayer asserted[7] that a self-contradictory description is not applicable to anything, and hence not to appearances as well. In spite of this, Bradley's point can be consistently interpreted as follows.

If something is red whenever we look at it, we may conclude that it is possibly red, but if it is red whenever we look at it with the left eye, green whenever we look at it with the right eye, and blue whenever we look at it with both eyes, we may conclude only that it looks red, green, and blue at the same time (different persons can simultaneously look at it in different ways), but we must not conclude that it is possibly red, green, and blue at the same time. Any particular description is consistent in itself, but all the descriptions taken together are inconsistent. The object we are describing may seem different, in accordance with the selected descriptions, but cannot be such per se. If we are really obliged – as Bradley believes we are – to describe motion in various ways which are inconsistent among themselves, we may say that though it appears as something real it cannot be real.

The main question is, however, whether, if we accept that motion exists, we are really obliged to accept the paradoxical $\underline{C}$. I am sure that most people would not like to proclaim the entire world of change a mere appearance, and this cannot satisfy us particularly in those instances – such as the presently considered – where we make that proclamation in view of our inability to find a better solution to our difficulties.

II. $\boxed{\neg\Diamond \underline{P}_1}$

Historically, there are two known strategies in denying $\Diamond \underline{P}_1$ without questioning the general hypothesis concerning the existence of plurality and motion (so also not denying $\Diamond \underline{P}_2$): atomism and radical empiricism.

1. Atomism

One of Zeno's arguments against plurality[8] is parallel to The Achilles in that it states that the hypothesis concerning the divisibility of bodies leads to the conclusion, which is assumed to be paradoxical for the reasons similar to those against $\underline{C}$ (§1), that a limited body can consist of an infinite number of constituent parts. Early Greek atomists used this argument of Zeno as an apagogic proof – not for the impossibility of any division, but for the necessity of the existence of atoms[9].

But, according to Leucippus and Democritus themselves, motion takes place in empty space[10], and hence there is nothing in the teaching of the physical atomists that would prevent Achilles from moving according to $\underline{P}_1$.

Beside atoms of the early Greek atomism, Epicurus introduced, however, by analogy (τῇ ἀναλογίᾳ[11]), minima which are not only impenetrable, but indivisible in every sense. He described them as 'that what is smallest within the atom', what is, as such, an absolute measure of everything larger[12]. A limited body as well as any limited part of empty space, contain necessarily a definite finite number of such minima[13], and this number is a priori fixed in each particular case.

These spatial minima we shall simply call topons, and this aspect of Epicurus' teaching – geometric atomism, because topons are not only physically but also geometrically indivisible. Epicurus introduced temporal minima too[14], which we shall call chronons, but we shall reserve the name kinematic atomism for later doctrines which contain only temporal, not physical and spatial minima.

Epicurean geometric atomism implies that the premise $\underline{P}_1$ in our version of The Archilles is not possible: according to $\underline{P}_1$ there is an $\underline{n}$ such that distance $[(2^n-1)/2^n, (2^{n+1}-1)/2^{n+1}]$ is smaller than a topon, which is absurd. This is sufficient, though we can also use the fact that according to Epicurus chronons exist, too, for the reductio ad absurdum of $\Diamond \underline{P}_1$.

There are many problems in geometric atomism. A physical atom can be of any shape, due to the fact that it is indivisible, and – according to early Greek atomism – ontologically characterized as an absolute unit by virtue of its impenetrability only. The geometrical atom, on the other hand, can hardly be of any shape, if it has to be absolutely, that is also geometrically, indivisible, and it is not at all clear – as it is not a geometrical point – how it can have size without being of any shape. It can be shown that the entirety of geometry breaks down if geometric atomism is true, but we shall not dwell on this point. I shall present here only the famous *reducto ad absurdum* of geometric atomism known as The French Interpretion of Zeno's fourth kinematic aporia[15].

Let all of the A's, just as all of the B's and all of the C's, occupy neighbouring *topons*, being in positions represented by the following diagram:

```
        A A A A
    B B B B →
          ← C C C C
```

Let the B's and the C's be moving in directions indicated by the arrows, while the A's are at rest. The stationary A's and the moving B's give

```
        A A A A
      B B B B
```

as the first possible next position. Similarly, the stationary A's and the moving C's give

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        A A A A
          C C C C
```

as the first possible next position. This means that the first possible next position of the A's, the B's, and the C's is

```
        A A A A
      B B B B
          C C C C
```

which means that position

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        B B B B
              C C C C
```

of the B's and the D's has been skipped. But why cannot the B's and the C's find themselves in that position?

The French Interpretation stops at this point: the geometric atomists do not seem to have a satisfactory answer to the last question. But we can go a little bit further.

Suppose that the geometric atomists' reply is that the *B*'s and the *C*'s cannot be in position

```
B B B B
      C C C C
```

just because there are no *topons* making that position possible. What would, however, happen if some *D*'s are moving parallel to the *C*'s according to the following diagram?

```
        A A A A
B B B B→ ←D D D D
      ← C C C C
```

The *B*'s and the *D*'s will collide when *B*'s and *C*'s are in position

```
B B B B
      C C C C
```

that is, the *B*'s and the *D*'s will collide at the mid-point of the allegedly indivisible *topon* (after a half of the allegedly indivisible *chronon*).

Trying to solve Zenonean problems and, at the same time, avoiding the specific difficulties of physical and geometric atomisms, the twentieth century atomists rely solely on kinematic indivisibles, either based on the existence of *time* indivisibles, or on the necessity of the existence, in any particular case, of the shortest not *actually* divided and only *a posteriori* indivisible phase, or phases, of a given change[16].

Kinematic atomism's solution of our version of *The Achilles* which is based solely on time indivisibles states that after sufficiently great *n* Achilles can neither move from $(2^n-1)/2^n$ to $(2^{n+1}-1)/2^{n+1}$, nor rest for $(2^{n+1}-1)/2^{n+1}$ h, because the period of $(2^{n+1}-1)/2^{n+1}$ h is shorter than a *chronon*.

This version of kinematic atomism can be reduced to absurdity in a way similar to that used to reduce geometric atomism to absurdity.

Let the *A*'s, the *B*'s, and the *D*'s be in the position presented in the following diagram:

```
        A A A A
B B B B→←D D D D
```

The *A*'s are at rest, while the *B*'s, and the *D*'s are moving as indicated.

If the *B*'s need one *chronon* to reach position

```
  A A A A
B B B B
```

and the *D*'s also need one *chronon* to reach position

```
A A A A
    D D D D
```

the *B*'s and the *D*'s will collide after a half of the *chronon*, which is absurd.

If the solution of our version of *The Achilles* is to be based on the necessity of the existence of the shortest not actually divided phase, or phases, of a given change – but in such a way that a transition from

```
    A A A A
B B B B
        C C C C
```

to

```
  A A A A
B B B B
  C C C C
```

though not being *actually* mediated by

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B B B B
      C C C C
```

does not imply that it *had to* happen during an indivisible period of time – this would not be a solution at all, because it does not exclude that the *B*'s and the *C*'s *too could have been* positioned thus:

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B B B B
      C C C C
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There is nothing to imply that Achilles *could not move* according to *P_1*, even if his normal motion is actually undivided, or consists of a finite number of not in actuality divided phases.

Neither geometric, nor kinematic atomism *per se*, or any combination of the two, can make *P_1* impossible without exposing itself to unsurmountable difficulties.

2. Radical Empiricism

When dividing a given body, we shall reach a point where no further division is possible – if the parts we get by division are to be perceivable. Berkeley, the most radical empiricist, used this fact to destroy the idea of the infinite divisibility of matter – if its esse est percipii: the fact that there are perceptive minima necessarily puts limitations on any division.

Contemporary radical empiricists, David Hilbert and Max Black for instance, who do not want to rely on the esse est percipii principle, appeal to ordinary language and to scientific practice in order to derive a Berkeleian conclusion concerning infinity.

If we do not want to violate the limitation on the scale of applicability of the word 'jump', we shall never speak of a 'thousandth inch jump', just as we shall never speak of a multicoloured space of a billionth part of a pin, if we want not to violate the limitation imposed by scientific knowledge about the nature of colour[17].

How do we come to believe that it is possible for a physical body to consist of an infinite number of physically heterogeneous parts, and that it is possible for Achilles to move according to P_1? We discover that physical bodies have concealed parts with concealed properties that are not evident to direct inspection but are progressively revealed by other means. Then we generalize, we assume that what is true of a macroscopic body is true of any of its parts, namely, that every concealed part has concealed parts with concealed properties. By such a generalization, we in fact make an extrapolation from a given experience of macro-level phenomena to a possible experience of micro-level phenomena[18], overlooking, or ignoring, the fact that there are limitations of scale upon the applicability of the words we need to describe micro-level phenomena.

Hence, without using the esse est percipii principle, contemporary empiricists show that it is not certain that it is possible for a physical body to consist of an infinite number of physically heterogeneous parts, and for Achilles to move according to P_1.

However, although we are shown that it may not be possible for a physical body to consist of an infinite number of physically heterogeneous parts, and for a motion to consist of an infinite number of staccato movements, we are not shown that it must be impossible for any physical body to consist of an infinite number of physically heterogeneous parts, and for any runner to move according to P_1. Even if it is true that there are limitations of applicability of words referring to known physical properties, this does not imply that it also must be true of every concealed property which could possibly be revealed, and of every possible runner.

If we take Achilles to be Zeus, P_1 *holds possible*: in moving according to the rhythm defined in P_1, Zeus could be, by definition, not exposed to any alleged empirical limitation. In consequence, we have been offered no solution to our problem.

III. $\boxed{\Diamond \underline{C}}$

There are two variants of *infinitism*, the doctrine which takes C to be possible: infinitism *without* infinitesimals and infinitism *with* infinitesimals.

1. Infinitism Without Infinitesimals

The infinitists who do not use infinitesimals[19] proceed from the statement that according to standard analysis there is enough time and enough room for an infinite number of discrete successive acts, such as Achilles' movements described in P_1: according to standard analysis, Achilles' path AB *metrically equals* the sum of $\aleph_0$ space intervals $[(2^n-1)/2^n, (2^{n+1}-1)/2^{n+1}]$ which he covers by his movements, and, similarly, 2 h *equal* the corresponding sum of time intervals, $1 + 1/2 + \ldots + 1/2^n + \ldots$ $(n=0, 1, 2, \ldots)$, during which Achilles makes his movements from $(2^n-1)/2^n$ to $(2^{n+1}-1)/2^{n+1}$ and rests after each of them.

After stating that according to standard analysis there is enough time and enough room for an infinite number of Achilles' movements described in P_1, the infinitists claim that for Achilles to perform an infinite number of acts described in P_1 it is enough to 'start and not stop after any finite number'[20]. In such a way Achilles will allegedly perform an infinite number of movements, in spite of the fact that the finiteness of the number of his movements is recursively preserved from movement to movement. Finally, the infinitists accept that Achilles can complete the series of an infinite number of strictly discrete successive acts of non-infinitesimal duration *without* having performed a *last* act.

To begin with the infinitists first statement: how is it possible that we find that according to standard analysis *there is not* enough time for an infinite number of Achilles' movements described in P_1, while the infinitists find that according to the same mathematical analysis *there is* enough time for an infinite number of these movements?

In our reasoning in §1, we stressed the fact that for any instant (any point) within the period of 2 h (within AB), it holds that up to that instant (that point) Achilles has performed a finite number of movements.

In their reasoning, the infinitists, however, stress the fact that no finite number can supply an adequate answer to the question of the number of

Achilles' rest-states when the 2 h have elapsed: after any finite number of performed movements, it holds that there is enough time within the 2 h for more movements to be made according to the rhythm defined in $\underline{P}_1$. In addition, they may say that after any finite number of rest-states Achilles is somewhere between A and B: $\forall n \exists \varepsilon ((2^n - 1)/2^n < AB - \varepsilon)$ (where $\underline{n}$ is finite). From that the infinitists draw the conclusion that after the two hours have elapsed the statement that Achilles has performed an infinite number of movements must be true; thus $\diamond \underline{C}$.

The infinitists do not locate the possible mistake in the reasoning to the conclusion that there is neither enough time nor enough room for an infinite number of movements performed by Achilles in 2 h according to $\underline{P}_1$. They only show a thing which we did not deny in §1, that, namely, within the period of 2 h and within AB there is always enough time and enough room for <u>more</u> movements to be performed – independently of however great the number of the movements which have been already performed is. And <u>from this</u> they infer that the 2 h period and distance AB are sufficiently large for an <u>infinite</u> number of performed movements.

Where we encounter a problem the infinitists simply draw an 'infinitary' conclusion by stressing some facts and ignoring others. Our difficulties lie in the fact that no finite number seems to be an adequate answer to the question concerning the number of Achilles' rest-states when the 2 h have elapsed; yet at the same time, there is not enough time for an infinite number of them to have occurred. The infinitists take the fact that no finite number seems adequate as a sufficient ground for the conclusion that there is enough time and enough room for an infinite number of performed acts.

When dealing with the problem of the recursively preserved finiteness of the number of Achilles' movements, the infinitists use the same strategy; commencing from the fact that no finite number can supply an adequate answer to the question concerning the number of Achilles' rest-states when the 2 h have elapsed and the alleged truth that 2 h are enough time for an infinite number of performed movements, they simply infer that Achilles will have performed an infinite number of movements defined by $\underline{P}_1$ if he does not stop after any finite number; and they do not show how Achilles will free himself from the situation of having performed a finite number of movements, and how he 'jumps' into the situation of having performed an infinite number of them.

To illustrate how far the infinitists have to go in their strategy to derive conclusions while ignoring problems, we shall dwell a little bit longer on the discussion concerning the third difficulty from §1, namely, the problem of the 'last act' in the performing of the so-called <u>super-task</u>[21], the task of completing the series of an infinite number of acts.

Let us consider the 'game with a marble' played by Black's 'transferring machines' 'Beta' and 'Gamma', which Black 'invented' in order to show that it is impossible to perform a <u>super-task</u>[22]. Let there be a marble in a left-hand tray, which some device, named 'Beta', transfers to an empty right-hand tray in one minute; after that, 'Beta' rests for one minute, while 'Gamma', a similar device, transfers the same marble to the left-hand tray; 'Beta' then transfers the marble to the right-hand tray in half a minute, and rests for one half of a minute while 'Gamma' transfers the marble to the left; and so on, 'Beta' always moves the marble from the left to the right while 'Gamma' is at rest, and 'Gamma' moves the marble from the right to the left while 'Beta' is at rest, the periods of the transfer of the marble from the left to the right and back decreasing according to a geometric progression converging to zero. Where is the marble at the end of exactly four minutes, when the machines come to a halt?

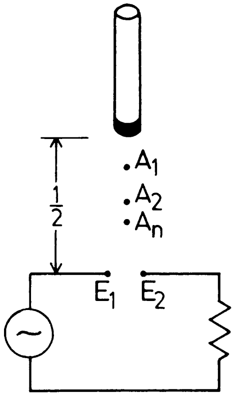
A similar super-task is described by Thomson[23]. Thomson's lamp is a reading lamp equipped with a button which, if pressed, switches the lamp on when it is off and switches it off when it is on, where the button, in addition, may be pressed in such a way that the first jab requires 1/2 of a minute, the second 1/4 of a minute, and so on according to a geometric progression. If the lamp operates thus, is it on or off when one minute has elapsed and we have stopped pressing the button?

Paul Benacerraf argues[24] that according to the conditions given in the description of the operation of Thomson's lamp nothing implies what the state of the lamp is when one minute has elapsed: the lamp can be broken at the instant when the first minute elapses and the second minute begins, as well as it can be on or can be off after that; the given description only concerns the period of time delineated by the one minute. Similarly, the marble transferred from the right to the left and vice versa within the period of four minutes can be broken at the beginning of the fifth minute, just as it can fall out of the trays.

Adolf Grünbaum cites[25], however, the additional conditions which have to be met in order to enable us to say where the marble is, and whether the lamp is on or off - after the completion of the infinite processes described by Black and Thompson.

Let us assume that not only the time intervals of marble transfer but also the distances through which the marble transfers have to be effected decrease in proportion to the available successive times, converging to zero by the time the four minutes elapse. If the marble is at rest after that, it is, according to Grünbaum, on the line of contact of the left and the right trays, after being transferred an infinite number of times from one tray to another.

The additional conditions concerning the Thomson lamp super-task are the following. At time t_0, when the lamp is off, the initial vertical distance between the button base and the horizontal circuit-opening $E_1 E_2$ is 1/2 (see the diagram).



Then after the base of the button has been depressed once to close the circuit, let it be raised after each such depression not all the way to its initial position A but to intermediate points A_1, A_2, A_3, ... , A_n,... , whose respective distances from $E_1 E_2$ are 1/8, 1/32, 1/128, ..., $1/2^{2n+1}$, ..., corresponding to the same sequence of time intervals which the button needs to reach from $E_1 E_2$ to A_1, A_2,..., A_n,... respectively (and from A_1, A_2,..., A_n...to $E_1 E_2$). If these conditions are met, then, Grünbaum claims, after an infinite number of circuit-closing jabs, at the end of one minute, at time $\underline{t_1}$, the state of the circuit is <u>predictably closed</u>, meaning that the <u>lamp will be on</u> thereafter, if the lamp circuit still exists intact at that and after that time.

One could say that, in order to reduce to absurdity Grünbaum's claim concerning the position of the marble after four minutes, it is enough to ask him whether the marble, being allegedly on the line of contact of the trays, has come to this position from the left or from the right. If it has come from the left, 'Beta' has moved it there, and 'Beta' needed a certain period of time to do that. If the marble has come to its last position from the right, 'Gamma' has moved it there, and 'Gamma' needed a certain period of time to do it. Whatever is the case, and however small the period of the last movement is, 'Beta' and 'Gamma' did not have enough time – from the beginning up to that last movement – to transfer the marble from the left to the right and back more than a finite number of times. But, Grünbaum would reply, <u>no last transfer took place at all</u>[26]. The marble has reached the last position without a last transfer from the right or from the left, that is, without arriving at its last position from anywhere!

Instead of explaining what this could possibly mean, to arrive at the last position from nowhere, Grünbaum uses a definitional <u>fiat</u>: in the sequence of marble transfers there is no last member.

Everything becomes even more curious when we turn to Thomson's process. Of course, Grünbaum claims that the jabbing motions issue in an on-state at time $\underline{t_1}$, <u>though</u> that on-state is <u>not</u> the terminus of any <u>continuously downward</u>

jabbing motion of positive duration (during which the lamp would be off)[27]. But, in a slightly different switching arrangement, concocted by Allen Janis[28], where nothing is essentially changed from a kinematical point of view, being moreover parallel to 'Beta''s and 'Gamma''s game with the marble, Grünbaum proclaims the corresponding Thomson process to be impossible only because the lamp is, by such an arrangement, neither predictably on nor predictably off at t_1 (again due to the fact that there is no last continuously jabbing motion) and the lamp cannot be in a neither-on-nor-off state after t_1 if the lamp circuit still exists intact at that time.

Janis' arrangement is the following. Let the switch button be movable through space intervals converging to zero and divided by $E_1 E_2$ as a mid-point (see the diagram above); the lamp is on or off depending on whether the button has arrived at $E_1 E_2$ from above or from below respectively: if the button base is at $E_1 E_2$ at time t, the existence of an on-state of the circuit at t requires that the button base has reached $E_1 E_2$ from above at or before t, and the existence of an off-state at t, while the base is present at $E_1 E_2$, requires the base to have reached $E_1 E_2$ from below at or before t.

Grünbaum's previous considerations enable us to assert that at time t_1 the button base cannot have reached $E_1 E_2$ either by a continuous approach from above or by a continuous approach from below. Thus, if the lamp circuit still exists intact at t_1 and thereafter, the lamp is then neither on nor off, in spite of the fact that the button base is at $E_1 E_2$.

Instead of adopting this conclusion concerning Janis' switching arrangement as something that sheds doubt on his previous considerations on infinite processes, Grünbaum, once more, ignores the problem, leaving his previous conclusions intact: Achilles can reach his goal after an infinite number of stops, without a last movement; the marble can reach the line of contact of the trays after an infinite number of transfers without a last transfer; the lamp can be on at the beginning of the second minute and thereafter after an infinite number of circuit-closing jabs, without that on-state being the result of any continuously downward jabbing motion of positive duration - under the previously defined conditions; but, the similar Thomson process is impossible under essentially the same conditions, viewed kinematically, only because by Janis' arrangement the non-existence of the last downward or upward jabbing motion implies the neither-on-nor-off state of the lamp by the end of one minute and thereafter. Thus, if there is no witness to the curiousness of the non-existence of the last act in completing the series of an infinite number of acts - as in the case of the marble transfers - the completion of the infinite process ought to be possible; if, however, such a witness does exist - the completion of essentially the same infinite process is impossible!

2. Infinitism With Infinitesimals

One of the difficulties with C (§1) consists of the conclusion that there is neither enough time within the period of 2 h nor enough room within AB for an infinite number of Achilles' movements defined in $\underline{P}_1$. The reason for such a conclusion is based on the fact that for any instant, however close to the end-instant of the period of 2 h, just as for any point, however close to the right end-point of AB, it holds that up to that instant, and up to that point, Achilles has performed a finite number of movements.

According to non-standard analysis, however, where the so-called Archimedes' axiom does not hold[29], there would be an instant for which we could not demonstrate that if Achilles has moved according to $\underline{P}_1$ up to that instant, then he has performed only a finite number of acts. Such an instant is any instant which is infinitely close to end-instant of the period of 2 h. Similarly, there is a point, and such a point is any point within AB infinitely close to B, for which we could not demonstrate that before reaching it, or passing through it, if Achilles has moved according to $\underline{P}_1$, he has performed only a finite number of movements.

Now, according to one version of the formula dating from the middle-ages[30], a finite distance cannot be <u>exhausted</u> by an infinite number of subdistances if there are not infinitesimals among them, while according to an other version of the same formula, a finite distance cannot <u>contain</u> an infinite number of subdistances of non-infinitesimal magnitudes. The similar holds for a limited period of time and its subintervals.

Putting aside the additional problems which the theory of infinitesimals introduces, we shall point out that even in the formally coherent version of the theory[31] the introduction of infinitesimals does not succeed in <u>shifting</u> the original problem.

According to non-standard analysis, there is a 'gap' between the members of the sequence of Achilles' rest-states $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_n$, ... with finite subscripts and the members of that sequence with transfinite subscripts. There also exists no greatest finite subscript number in that sequence, and the number of members with finite subscript numbers is not finite. So, we are faced with the problem of Achilles' 'bridging the gap' between the positions with finite and the positions with transfinite subscripts, that is, with the problem of his 'jumping' into a position infinitely close to B. Independently of this problem, Achilles has to be in an infinite number of positions with finite subscripts, and the finiteness of the number of his rest-states is recursively preserved. And again, independently of the existence of infinitely small space distances, and infinitely short time intervals, there is neither enough room nor enough time for an infinite number of Achilles' rest-states $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n, \ldots$ with <u>finite</u> subscripts.

IV. $\neg\Diamond\underline{C}$, Without the Exclusion of $\Diamond\underline{P}_1$ and $\Diamond\underline{P}_2$ and With No Explanation as to What's Wrong With $\Diamond\underline{P}_1$, $\Diamond\underline{P}_2 \vdash \Diamond\underline{C}$

Under this heading, we shall consider two strategies of denying $\Diamond\underline{C}$, which are successful in dealing with some versions of The Achilles and some related aporias, but which strategies, in effect, do not exclude $\Diamond\underline{P}_1$ and $\Diamond\underline{P}_2$, yielding no answer to the question of the number of Achilles' rest-states when we assume $\underline{P}_1$ and $\underline{P}_2$ to be true.

1. Finitism

According to Geach, who subscribes to Frege's analysis of identity statements[32], the identity is always relative in the following sense: When one says "$\underline{x}$ is identical with $\underline{y}$", this...is an incomplete expression; it is short for "$\underline{x}$ is the same $\underline{A}$ as $\underline{y}$", where '$\underline{A}$' represents some count noun understood from the context of utterance...'[33] Thus $\underline{x} \equiv \underline{x}$ is also meaningful only if in the given context $\underline{x}$ refers to a thing of a certain kind; for instance, $\underline{x}$ cannot refer to a part as such: 'part as such' cannot be a sortal.

On this reasoning, David Schwayder, and Max Black in his first analysis of Achilles and the Tortoise[34], tried to solve Zeno's metric and kinematic aporias: Achilles' path contains necessarily a finite number of parts properly identified, it is also finite measured by any means, and Achilles, in order to catch up with the tortoise, is not called upon to do the logically impossible, to complete the series of an infinite number of acts.

To count parts is possible only if one uses some count noun to single them out. If we, however, single them out by using proper sortals, we shall necessarily obtain a finite number of concerning parts: a finite number of pebbles, a finite number of blades of grass, a finite number of atoms, and so on [35].

Similarly, a limited body is metrically finite measured by any means. Any unit measure represents the size of something physical, however small. Be it a pebble, or anything smaller than pebble, it is contained a finite number of times in Achilles' path. This truth is untouched by the fact that the number of ways that may be used to define the unit measure is not limited[36].

In no case it is possible that a limited body have been measured by an infinitely changing measure unit. If Achilles' progress in chasing the tortoise were really such a measurement of the path that remains to be covered to the goal, he couldn't reach up with the tortoise, but, as Black put it, 'Achilles is not called upon to do the logically impossible', but we are those who create 'the illusion of the infinite task by the kind of mathematics that we use to describe space, time, and motion'[37].

It may be said that by accepting the finitists conclusions we can truly solve the version of *The Achilles* where Achilles moves *legato*. Moving normally, Achilles will reach the goal after a finite number of steps, heart beats, deep breaths, and so on, while the geometric progression which we use to describe his motion refers to nothing that he really does.

What, however, happens if Achilles moves *staccato*, according to $\underline{P}_1$? He then *does move* in accordance with a geometric progression, and its members *do refer* to something which Achilles *actually does*. By moving in this way, Achilles is *really measuring* his path with a unit which changes from step to step, being always recursively defined in terms of the previous step. If the number of members of that geometric progression is not finite, and if it is not possible to obtain the result of the measurement by using an infinitely changing unit, does this mean that Achilles will not reach his goal? And if he doesn't reach the goal, where will he be when the 2 h elapse? What is the answer to the question of the number of his actually differentiated movements?

There is nothing in the finitists' teaching which would make Achilles' movement according to $\underline{P}_1$ impossible. If he really does move according to $\underline{P}_1$, and if the number of members of the geometric progression is not finite, then it seems inevitable that the number of Achilles' movements, and rest-states, cannot be finite when the 2 h elapse; there is no explanation which would prevent us to derive $\Diamond\underline{C}$ from $\Diamond\underline{P}_1$ and $\Diamond\underline{P}_2$, and, in particular, the question about the number of Achilles' rest-states if the 2 h have elapsed has not been answered.

2. Indefinitism

Whether something is one single thing, in spite of the fact that it has many parts, or a set consisting of many things, is, *up to a certain point*, a matter of *linguistic convention*. This is Aristotle's first reply to the so-called aporia concerning the whole and the parts raised in the struggle between the monists and the pluralists[38]. The existence, oneness, and multitude can be spoken of in multiple ways (πολλαχῶς λεγόμενα) and in certain cases we can alternately favour the oneness and the multitude.

However, there are cases where, in accordance with Aristotle's later teaching on *primary meanings*[39], we are dealing with a *single thing* that is *only potentially something multiple*.

A thing that has no heterogeneous parts, and also neither discrete or contiguously touching parts[40] – is one single thing in the primary sense, because it is only potentially multiple. Positively expressed, something that is homogeneous and continuous in itself[41] must be characterized as truly one single thing, because in reality it is in no sense multiple, due to the fact that its alleged parts would have been its parts only if they were actualized by an appropriate division; though the existence, oneness, and multitude can be spoken of in multiple ways, their primary meaning excludes the alternative of such a thing being truly multiple.

Achilles' path is really multiple insofar as it consists of a number of heterogeneous, discrete, or contiguously touching parts. Can the number of such parts be infinite?

Aristotle subscribes to atomists' point that Zeno's arguments against plurality show that we can never, by any division, successive or simultaneous, even in principle, actualize an infinite number of potential parts[42].

Aristotle, however, does not reach the atomistic conclusion that physical or spatial indivisibles must exist, he admits to nothing more than that by any division we cannot obtain an infinite number of parts[43].

If an infinite division is not possible, and at the same time there are no indivisibles, this means that the number of parts we could obtain by a division, successive or simultaneous, is a finite, but not a definite finite number. Though any particular successive division is carried out only up to a certain point, we always get a greater number of parts by continuing the division. Similarly, even though any particular simultaneous division can result in only a finite number of parts, a new simultaneous division that would yield a greater number of parts is always possible. In both cases – in the case of successive as well as in the case of simultaneous division – an infinite number of parts is never obtained, even though there is no greatest number of parts obtainable by division. This is the essence of indefinitism[44].

What holds for the path, also holds for Achilles' motion itself. Because 'oneness' has a variety of meanings, 'one single movement' is an ambiguous term[45]. A movement can be said to be one single movement beause it is continuous in itself, that is, because it is a legato motion. But this same movement can be said to be multiple if it consists of phases which are in form (κατ' εἶδος) different among themselves: the speed and the manner of its variation can change during the motion, in which case the motion is non-uniform (ἀνώμαλος)[46]. If, however, the speed and its variation are constant and uniform respectively during the whole legato motion, such a motion constitutes one single movement in the primary sense[47].

Is it possible for Achilles' motion within a finite time to consist of an infinite number of actually differing phases? According to Aristotle, this is not possible for the reasons analogous to the reasons why Achilles' path cannot consist of an infinite number of parts. Hence $\neg\Diamond\underline{C}$.

$\underline{P}_1$ is, however, possible, because there are neither space indivisibles nor time indivisibles. So, if we accept the indefinitist conclusions, we can solve the problems concerning the legato version of The Achilles, but this does not help us with our version of it. When the 2 h elapse, that is, when $\underline{P}_2$ becomes true, then, on the one hand, we should not claim $\underline{C}$ true, because allegedly $\neg\Diamond\underline{C}$, and, on the other hand, we cannot cite a finite number as the number of Achilles' rest-states, because for any finite number it holds that Achilles has had to be in more rest-states according to $\underline{P}_1$, and $\underline{P}_1$ is possible. To state that the number of rest-states is finite but not definite makes, however, no sense when we speak of Achilles' actually performed differentiated movements.

§3. DISCUSSION

Bearing in mind our version of the problem (§1), we have seen that we cannot be satisfied by any of the considered proposed solutions of The Achilles (§2) for diverse reasons.

The price of the proposal put forward by the negative dialecticians is too high: be the world of change real or be it an appearance, we would like to speak of it in a consistent way. However, the essence of their proposal is that this is impossible. We shall accept their proposal only as a last resort.

The atomistic versions which allegedly solve the problem by making $\underline{P}_1$ impossible have been reduced to absurdity - not to mention other difficulties. True, the reductio ad absurdum can be avoided by additional restrictions, but they are inadmissibly ad hoc: BBBB and DDDD (see diagrams above) shouldn't start moving at all, if they are going to meet at half topon, or after half a chronon.

The restrictions imposed by radical empiricists seem unjustifiably limiting: nothing we know about the empirical world justifies the conclusion that it must be impossible for anybody to move according to $\underline{P}_1$. We must also have a solution for the case that such a movement is possible.

The infinitists' strategy of using a definitional fiat to 'kill' the problem wherever it arises can hardly convince us that $\Diamond\underline{C}$. Even if we accept that when viewed mathematically it is meaningful to speak of $\aleph_0$ finite distances $[(2^n-1)/2^n, (2^{n+1}-1)/2^{n+1}]$ which cover the whole distance between zero and 1 without any such distance being the last one on the right, we can hardly accept that a marble can reach its final position after being transferred an infinite number of times from the left to the right and vice versa, arriving there from nowhere. The necessity of 'killing' the witness introduced in a Thomsonean process that is quite analogous to Black's process in order to make that Thomsonean process possible, shows us most characteristically how far the infinitists are forced to go in using the method of definitional fiat. - The infinitists who use the infinitesimals do not shift the original problem at all - let aside the additional difficulties.

The finitist and the indefinitist teachings which stress the importance of the manner of differentiating parts and distinguishing between real and potential parts seem quite acceptable in implying the possibility of $\underline{P}_1$ and $\underline{P}_2$ to be true while asserting $\underline{C}$ to be impossible, but, not faced with, or ignoring, the staccato version of The Achilles, neither the finitists nor the indefinitists offer an adequate answer to the question about the number of Achilles' rest-states in the case that $\underline{P}_1$ and $\underline{P}_2$ are true.

Looking at all these deficiencies of the offered proposals, we may try to intuit the 'ideal' solution to our problem. The 'ideal' solution would be a solution which would

a) preserve the consistency of the world of change;

b) introduce no indivisibles of any kind;

c) impose no radical empiricist restrictions which would make $\underline{P}_1$ impossible;

d) justifiably accept $\neg\Diamond\underline{C}$ after explaining why $\Diamond\underline{P}_1$ with $\Diamond\underline{P}_2$ do not entail $\Diamond\underline{C}$.

In order to formulate such a solution, we shall now consider the acceptable aspects of the forwarded proposals.

The merit of the negative dialectic, from ancient times to today, lies in indicating the difficulties and deficiencies of the forwarded or possible proposals of the solution to Zenonean paradoxes: if the observed deficiencies cannot be resolved within the proposed solutions, the result of Zeno's dialectic should be accepted at least in its sceptical aspects: we have to admit that we are unable to speak of the world of change in a consistent way. It may be that in order to preserve that world we have to make peace with the inevitable incompatibilities between the world described in the Zenonean premises that make the goal unreachable and the everyday world in which Achilles has no difficulties in catching up with the tortoise.

The merit of atomism, i.e. of its variants, lies in the insistance that by the acceptance of the infinite division of Achilles' movement his goal remains unreachable.

The unacceptable conclusion based on $\underline{P}_1$ and $\underline{P}_2$ asserted in infinitism without infinitesimals necessitate our avoiding the admission that the conjunction of $\underline{P}_1$ and $\underline{P}_2$ can ever be true, and this represents the foremost merit of that variant of infinitism.

The merit of the infinitism with infinitesimals lies in its negative implication, which states that a finite distance, or a finite period of time, cannot contain an infinite number of subintervals if all of them are non-infinitesimals.

The merit of finitism lies in its stressing that measured by any appropriately predefined unit a limited body, or a limited period of time, always renders a finite result.

The merit of indefinitism consists in its restrictions imposed on the formulation of the problem itself, related to the difference between real and potential parts: we are faced with a real problem only when it is defined in terms of Achilles' actually differentiated movements, such as defined by the description of his staccato motion.

Armed with the acceptable insights of the proposals as well as with our experiences of their deficiencies, we shall now sketch a new solution of the problem.

§4. SOLUTION

Looking formally at our classification of the proposals for the solution to The Achilles, one may find that it is of note that – with the sole exception of the negative dialecticians, who deny the very possibility of the existence of the world of change – nobody has questioned the possibility of $\underline{P}_2$. Of course, considering the content of $\underline{P}_2$, one can easily discover why this is so: $\underline{P}_2$ only implies that the 2 h of Achilles' movement will elapse and says that when 2 h elapse Achilles is definitely at rest – all this seeming trivially possible. The fact, however, that not only has nobody questioned $\Diamond\underline{P}_2$, but that nobody has even considered the possibility of $\neg\Diamond\underline{P}_2$, suggests that everybody – pace negative dialecticians – has taken the unconditional possibility of $\underline{P}_2$ – $\Box\Diamond\underline{P}_2$ – as an evident truth. But, is what $\underline{P}_2$ implies really unconditionally possible?

Assuming that Achilles' staccato motion is a way to measuring distance AB, accepting also that after any finite number of movements Achilles can move once more in accordance with $\underline{P}_1$ without reaching B, and taking as true that it is impossible for a finite distance to have been measured by an infinitely changing unit, we are unexpectedly led to conclude that, though generally possible, $\underline{P}_2$ is not possible if $\underline{P}_1$ is true; $\underline{P}_1$ being true, the 2 h cannot elapse, and they will not elapse! There is a world, and that is exactly the world described by $\underline{P}_1$, in which $\underline{P}_2$ is impossible; so $\neg\Box\Diamond\underline{P}_2$.

The conclusion that it is not unconditionally true that 2 h can elapse may look very odd, because it seems that *by simply waiting* we can show that 2 h must elapse. But would this really be a proof that 2 h *must* elapse? No! That would only prove that the 2 h *have elapsed*. Would 2 h elapse in Achilles' world when he moves according to P_1 ?

The lesson we have learnt through the acceptable aspect of radical empiricism can help us here. For *us*, who *do live* in the world where there are minima everywhere, and whose perceptive and apperceptive thresholds are comparatively unchangable, 2 h *must* elapse. But, in Achilles' world where movement takes place according to P_1 there are, *ex hypothesi*, no minima. Without minima, his 2 h is not a finite time.

The fact that 2 h have *de facto* elapsed can hardly *prove* that the world of Achilles' movement according to P_1 is impossible. But this *implies* that in the *particular* case, if 2 h *have* elapsed, Achilles *couldn't have* moved in accordance with P_1 during the *whole* of the 2 h. If those 2 h have elapsed, Achilles has had to stop moving according to P_1 somewhere.

If P_1 has been true, then, necessarily P_2 has not become true, and if P_2 is true, P_1 couldn't have been true. If the description of Achilles' movement given in P_1 has not become false, the 2 h haven't elapsed, and Achilles has performed a finite number of movements. If, however, P_2 has become true, this entails that Achilles has had to *stop moving* according to P_1 *somewhere* – where exactly is a matter of *empirical* fact.

If P_1 has not become false there is no question of the *last* movement by which Achilles has reached B, simply because Achilles cannot reach B until P_1 becomes false. If, on the other hand, Achilles has reached B, he has done it, necessarily, by performing a *last* movement in a series of a necessarily *finite* number of movements, since in order to reach B Achilles has had to stop moving according to P_1 (somewhere).

Let $F(\mathcal{A}_k)$ mean that Achilles has been in k rest–states. If P_1 is really possible – and we assume it is – then $\forall k \Diamond F(\mathcal{A}_k)$. But, $\forall k \Diamond F(\mathcal{A}_k)$ does not imply $\Diamond \forall k F(\mathcal{A}_k)$. One is led to the transition from the former to the latter by the addition of P_2, which is possible, to P_1, which is also possible. P_2 being assumed, however, *incotenable* with P_1, $\Diamond \forall k F(\mathcal{A}_k)$ can be avoided.

This resolution of the problem cannot be formally expressed in any modal logic system with the strict implication reducible to the material one. We need to state $\Diamond P_1$, $\Diamond P_2$, $\Box(P_1 \Rightarrow \neg P_2)$, and $\Box(P_2 \Rightarrow \neg P_1)$, but then, supposing P_1 and P_2 (being possible) to be true, we can't avoid a contradiction.

The solution can easily be expressed, however, with the aid of the *relevant* implication. In relevant logics, it doesn't hold that by the addition of a new hypothesis the hypothesis previously introduced must remain deducible[48]: $A, B \vdash A$ does not always hold, and, in particular, we mustn't continue to assert $\underline{P}_1$ – $\underline{P}_1$, $\underline{P}_2 \vdash \underline{P}_1$ – if added hypothesis $\underline{P}_2$ is <u>incotenable</u> with $\underline{P}_1$, the <u>incotenability</u> of $\underline{P}_1$ and $\underline{P}_2$ being defined as

$$\neg(\underline{P}_1 \circ \underline{P}_2) \stackrel{def}{=} (\underline{P}_1 \rightarrow \neg \underline{P}_2)$$

(where '$\rightarrow$' is relevant implication and '$\circ$' intensional conjunction reducible to it with the aid of the ordinary negation[49]).

If we adopt $\underline{P}_1$ and $\underline{P}_2$ as <u>possible</u> but <u>incotenable</u> premises, we can save the world of change without any of the deficiencies of the proposals offered so far, conserving all their acceptable aspects.

In principle, there are no indivisibles, but in both the everyday and scientific worlds the empirical constraints exclude situations in which $\underline{P}_2$ would not be possible, and so, <u>under these constraints</u>, $\neg\Diamond\underline{P}_1$. In a possible world where these constraints are removed, $\underline{P}_1$ is, however, possible, but, if movement is governed by $\underline{P}_1$, $\underline{P}_2$ cannot ever be true; thus, nevertheless $\neg\Diamond\underline{C}$.

In principle, the greatest number of Achilles' movements defined by $\underline{P}_1$ does not exist, but the fact that, if Achilles has moved according to $\underline{P}_1$, the 2 h could not have elapsed, excludes a situation where no finite number could supply an adequate answer to the question of the number of Achilles' rest–states, and that is in accordance with the fact that the finiteness of the number of Achilles' movements is recursively preserved. The fact that within the finite distance AB and the period of 2 h there is always enough room and enough time for a number of Achilles' movements greater than any given finite number <u>is</u> reconcilable with the fact that within that distance and that period of time there is neither enough room nor enough time for an infinite number of <u>performed</u> movements – if $\underline{P}_1$ and $\underline{P}_2$ are <u>incotenable</u>.

The fact that when we speak of subdistances and subintervals in a general, non–specified way, we cannot cite a definite finite number of subdistances of AB, and subintervals of the 2 h period, doesn't mean that the number of parts of a limited body and the number of phases of a change performed in a limited period of time are not definite finite numbers when referring to real parts and real phases: Achilles' path does consist of a definite finite number of physically heterogeneous parts, and Achilles' motion during the whole period of 2 h has had to consist of a definite finite number of performed differentiated movements, as those in Achilles' <u>staccato</u> run; the number of movements has not been fixed only within the time when it has been <u>becoming</u> greater and greater, namely <u>within</u> the 2 h, but all the same, at any given moment, it has been definite and finite.

One could try to reduce our claim that $\underline{P}_1$ and $\underline{P}_2$ are <u>incotenable</u> to absurdity by a thought experiment which would 'mix' the world of $\underline{P}_1$ and the everyday world in which $\underline{P}_2$ is to become true.

Let Achilles move not staccato but in a normal manner, legato, and at a constant speed with which he will reach B in 2 h – this being a more appropriate way of movement with regard to the fact that he is, after all, a man. Let Zeus, however, move parallel with him at double speed but staccato, covering the distances of Achilles' path and resting after each of them in the way described by P_1. On one hand, Achilles will reach B just as all of us would, which means that the 2 h will have elapsed. On the other, if movement according to P_1 is at all possible, Zeus, the immortal, will be able to move in this manner. He may be capricious, not wanting ever to stop moving in such a way. Where will Zeus be,and in how many rest-states will he have been, when Achilles reaches the goal?

According to our solution, the essential difference between this situation and the original one (from § 1) lies in the fact that now there are two runners, the one who belongs to the everyday world and for whom 2 h must elapse, and the other for whom, belonging to the world of P_1, 2 h cannot elapse. However, the two runners move in such a way that Zeus is never behind Achilles. Thus, it seems, either Zeus' 2 h must elapse if Achilles' 2 h must elapse, or Achilles' 2 h cannot elapse if Zeus' 2 h cannot elapse.

It may seem that the situation with two runners indicates that our solution is implausible.

However, first, formally speaking, there need not be a contradiction in the set of the following statements:

- it is possible that Zeus has moved according to the rhythm defined in P_1 (Zp_1);
- if it hasn't become false that Zeus has moved according to the rhythm defined in P_1 (Zp_1), then, necessarily, his 2 h have not elapsed ($\neg Z_{2h}$);
- if Zeus' 2 h have not elapsed ($\neg Z_{2h}$), then, necessarily, Achilles 2 h have not elapsed ($\neg Ach_{2h}$);
- if Achilles 2 h have elapsed (Ach_{2h}), then, necessarily, Zeus' 2 h have also elapsed (Z_{2h});
- Achilles 2 h have elapsed (Ach_{2h}).

A contradiction would result only if by defining necessary implications, which occur in these statements, the relevancy conditions are ignored or mistakenly treated. If we, however, assume Zp_1 and Ach_{2h} to be relevantly inconsistent, that is, incotenable [50], no contradiction can be derived from the set of the above statements.

And, second, we have found out why we are prone to believe that one of the two statements which we take as incotenable must unconditionally become true: we do live in Achilles' world where he moves legato, while we do not live in the world of Zeus who moves staccato. Living in Achilles' world, we can always easily wait for him to reach the goal, while we can never follow all of Zeus' movements in order to see whether Achilles will reach the goal.

This means that we can always easily witness Zeus' giving up his staccato motion, while we can never witness the situation in which the statement about Achilles' reaching the goal cannot become true because of Zeus' following the law of his staccato motion.

Let us suppose that Hera, the godess, waiting for her husband at B, spies what he is doing with Achilles. Following every one of Zeus' movements, she will, contrary to us, embrace neither Zeus nor Achilles. In order to meet her husband, she would have to stop spying on him: she would have to 'jump' into a world incompatible with Zeus' permanent staccato motion, leaving Achilles to reach the goal, because this fact would force her husband, by logical necessity, to give up his staccato dance with Achilles. Where exactly would he do it, is not a priori determinable. Where he gave up the dance, if Achilles has reached the goal, is a matter of empirical fact.

Achilles' world encompassing his reaching the goal is inaccessible from Zeus' staccato world, the two worlds being incomposable.

It may seem incorrect to say that Zeus, moving staccato, will never reach the goal, because in ordinary language 'x will never happen' entails 'x will not happen even if you wait more than 2 h'. But this entailment holds only in ordinary language, which is a suitable language for ordinary experiences. The experience of Hera who is spying on her husband is not such an experience, and by doing this, her 2 h will not elapse; two hours are not necessarily a finite time.

NOTES

1. See Plato, Parmenides, 128 A.
2. Cf. Plato, Timaeus, 38 A and Aristotle, Physics, 221 b 3-7ff.
3. See DK 82 B 3.
4. Cf. G.W.F. Hegel, Vorlesungen über die Geschichte der Philosophie II (Werke 19), p. 62, Suhrkamp Verlag, Frankfurt am Main, 1971.
5. See F.H. Bradley, Appearance and Reality - A Metaphysical Essay, ch. XVIII, Clarendon Press, Oxford, 1966.
6. Ibid., p. 181.
7. See A.J. Ayer, Metaphysics and Common Sense, p. 66, MacMillan, 1969.

8. DK 29 B 1.

9. Cf. Aristotle, On Coming-To-Be and Passing-Away, 316 a 15ff.

10. See ibid., 325 a 31.

11. Epicurus, To Herodotus 58

12. Ibid., 59
13. Ibid., 56-57

14. Ibid., 61-62

15. Cf. P. Tannery, Pour l'histoire de la science Hellène, Paris, 1887. F. Evelin, 'Le mouvement et les partisans des indivisibles', Revue de metaphysique et morale 1,1893, G. Noël, 'Le mouvement et les arguments de Zénon d'Elée', Revue de metaphysique et morale 1, 1893. - I don't think that this interpretation, followed by many scholars, is historically correct (see M. Arsenijević, Prostor, vreme, Zenon (Space, Time, Zeno), pp.91-92, Serbian Philosophical Society and Graphic Institute of Croatia, Belgrade-Zagreb, 1986), but, in any case, it can be used as an argument against geometric atomism.

16. I shall not discuss here Grunbaum's interpretation, according to which all of the famous kinematic atomists, W. James, A.N. Whitehead, and P. Weiss, allegedly side the kinematic atomism of the former kind (cf. A. Grünbaum, 'Relativity and the Atomicity of Becoming', Review of Metaphysics 2, band 4, 1950 and Modern Science and Zeno's Paradoxes, p. 48, George Allen and Unwin Ltd., London, 1968).

17. M. Black, Problems of Analysis, pp. 116-118, Cornell University Press, Ithaca, New York, 1954.

18. Berkeley was the first to claim that our faith in infinite divisibility has its origin in this extrapolation, based on the way in which the mathematicians proceed: whenever the geometer speaks of particular lines and figures, he 'considers them abstracting from their magnitude', which implies 'that he cares not what the particular magnitude is, whether great or small, but looks on that as a thing indifferent to the demonstration' (Berkeley, The Principles of Human Knowledge, §126, p. 154, George Routledge and Sons, London and New York, 1878). David Hilbert, a mathematician, was of the same opinion, and he tried to solve The Achilles on that basis (see D. Hilbert and P. Bernays, Grundlagen der Mathematik I, § 1 c 2, p. 16, Springer Verlag, Berlin, Heidelberg, New York, 1968).

19. Infinitism is the religion of mathematicians. Among philosophers who used this strategy in trying to solve Zeno's paradoxes the most influential are :
B. Russell (see Our Knowledge of the External World, lectures VI, VII, The Open Court Publishing Company, Chicago and London, 1914),
R. Taylor (see 'Mr. Black on Temporal Paradoxes', Analysis 12, 1951, and 'Mr. Wisdom on Temporal Paradoxes', Analysis 13, 1952),
A. Grünbaum (see Modern Science and Zeno's Paradoxes, ch.II, George Allen and Unwin Ltd., London,1968), and
W.C. Salmon (see Space, Time, and Motion, ch. II, Dickenson Publishing Co., Inc., Encino and Belmont, California, 1975).
20. J. Watling, 'The Sum of an Infinite Series', Analysis 13, 1952, p. 48.

21. Cf. J.F. Thomson, 'Tasks and Super-Tasks', Analysis 15, 1954.

22. See M. Black, 'Achilles and the Tortoise', Analysis 11, 1951, p. 97.

23. See above, n. 21.

24. P. Benacerraf, 'Tasks, Super-Tasks and Modern Eleatics', The Journal of Philosophy vol. LIX 24.

25. A. Grünbaum, Modern Science and Zeno's Paradoxes, pp. 94ff.

26. Ibid., p. 105.

27. Ibid., p. 100.

28. See ibid., p. 101.

29. Cf. A. Robinson, Non-Standard Analysis, pp. 55ff. and pp. 266-7, North-Holland Publishing Company, Amsterdam, 1970.

30. Cf. C.B. Boyer, The Concepts of the Calculus: A Critical and Historical Discussion of the Derivative and the Integral, ch. III, Dover Publications, New York and London, 1939.

31. The solution of The Achilles proposed by G.J. Whitrow in 1961 is in accordance with non-standard analysis, which came into being a few years later (cf. G.F. Whitrow, The Natural Philosophy of Time, p. 151, Thomas Nelson and Sons Ltd., 1961 and A. Robinson, Non-Standard Analysis, p. VIII and pp. 58ff.).

32. See P.T. Geach, Logic Matters, p. 238, Basil Blackwell, Oxford, 1972.

33. Ibid., loc. cit.

34. See M. Black, 'Achilles and the Tortoise', Analysis 11, 1951.

35. Cf. ibid., pp. 100–101.

36. Cf. D.S. Schwayder, 'Achilles Unbound', The Journal of Philosophy vol. LII 17, pp. 454ff.

37. M. Black, 'Achilles and the Tortoise', Analysis 11, 1951, p. 101.

38. See Aristotle, Physics, 185 b10ff.

39. Cf. Aristotle, The Nicomachean Ethics, 1096 b 27, and Metaphysics, 1003 a 21.

40. Two parts are contiguously touching if they are physically heterogeneous or if they are not physically grown together. For Aristotle's definition of contiguity see Physics, 227 a 6.

41. For Aristotle's definition of continuity see Physics, 227 a 10. That the distinction between contiguous and continuous touchings can be inclusive see M. Arsenijević, 'Dodirivanje' ('Touching'), Filozofske studije (Philosophical Studies) VI, 1975.

42. See Aristotle, On Coming-to-Be and Passing-Away, 317 a 8ff.

43. See Aristotle, Physics, 231 b 15.

44. For the justification of my use of name indefinitism for the theory under consideration cf. G.W. Leibniz, 'The Theory of Abstract Motion: Fundamental Principles', Philosophical Papers and Letters, p. 139, D. Reidel Publishing Company, Dordrecht-Holland and Boston-U.S.A., 2nd edition, 2nd printing, 1976, and Kant, Kritik der reinen Vernunft (Werke III), pp. 351ff., Druck und Verlag von Georg Reimer, Berlin, 1911.

45. Aristotle, Physics, 227 b 3.

46. Cf. ibid., 228 b 15ff.

47. In our century, H. Bergson proposed a solution to Zeno's Arrow completely based on Aristotle's analysis of the nature of the oneness of a motion (see H. Bergson, L'évolution creatrice, pp. 334ff., Librairie Felix Alcan, Paris, 1932).

48. See A.R. Anderson and N.D. Belnap, Jr., Entailment - The Logic of Relevance and Necessity, §§3, 4, Princeton University Press, Princeton and London, 1975.

49. Intensional conjuction was introduced by R.K. Meyer in 'Some Problems No Longer Open for E and Related Ligics', Journal of Symbolic Logic 35, 1970.

50. De facto, the term 'relevant consistency', used in relevant logics, and Nelson Goodman's neologism 'cotenability' are being used synonymously (cf. A.R. Anderson and N.D. Belnap, Jr., op. cit., pp. 345-6).