DIALEKTIK
1994/3
Enzyklopädische Zeitschrift für Philosophie und Wissenschaften

Naturalismus in der Philosophie der Mathematik?

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Mathematics, Infinity and the Physical World

ABSTRACT: *It is shown that geometric objects can supervene on the physical: one should let individual variables range over entities of the highest dimension and start with part-whole and touching relations as primitives. However, some geometric objects definable analytically, such as segments of the so-called remarkable curves, are shown to be impossibly existent in the physical world, although they are mathematically indistinguishable, in view of their existence, from those which are possibly existent. The cause of that is shown to lie in the mathematical treatment of infinity, which allows an infinite heterogeneity to appear at a higher-order level.*

> *"Les mathématiciens ont autant besoin d'être philosophes que les philosophes d'être mathématiciens"*
>
> Leibniz, *Letter to Malebranche*, March 1699

1. The setting of the problem

It would be hard even to cite all questions discussed, or still to be discussed, which concern the relation between any two of the three notions from the title of this paper. Fortunately, the combination of all three by itself makes the number of questions smaller, and the particular question I am going to deal with here will enable me, I hope, to introduce it unproblematically isolated from the rest of questions.

However, the answer to the question can still be expected to be also illuminating for some other aspects of the problem for the following reason. Taken generally, we are going to deal with the question of *whether*, *how* and *which* geometric entities can exist in the physical world. Now, on the one hand, geometry is probably more closely related to the

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physical world than any other branch of mathematics. On the other hand, however, the machinery of modern mathematics enables us to deal with geometry in a purely analytic manner, where no intuitions borrowed from the physical world are essentially needed. This curious position of geometry should make, after all, the answer to our question illuminating for a more general question about the relation between mathematics and the physical world as well as for the fact that there are geometric entities definable according to the mathematical way of proceeding which are not possibly existent in the physical world and which are yet, in view of their possible existence, mathematically indistinguishable from those which are possibly existent.

The concept of infinity, essentially involved in our central question, will appear later, when we start to investigate the so-called remarkable curves. But, before doing that, I want to clarify the sense in which the existence of geometric entities in the physical world can be unproblematically spoken of, by dispelling three critical remarks directed against the possibility of such an existence.

1. As the first point, I want to dispel the argument that there can be no geometric points, lines and surfaces in the physical world due to the fact that *physical* points, lines and surfaces are never to be said to be null-, one- and two-dimensional entities respectively, in a strict sense of the word.

Given that we agree that, strictly speaking, points, lines and surfaces which are said to be physical are necessarily not null-, one- and two-dimensional entities, this doesn't mean that there are no geometric entities in the physical world, simply because that which we have supposedly agreed upon doesn't imply that it is impossible to define null-, one- and two-dimensional entities *via* that which is physical *par excellence* and to show, on the basis of such a definition, that there can be geometric entities in the physical world.

Let us remember how points, lines and surfaces are conceived by Aristotle.[1] They are understood as limits of limited physical bodies, belonging to them without being parts of them, and if there is a physical cube, for instance, there must be also geometric points, lines and surfaces in the physical world.

By using the tools of formal logic, one can define *n*-dimensional entities as limits of (limited) (*n*+1)-dimensional entities by letting individual variables range over limited (*n*+1)-dimensional entities and starting with

part-whole and *touching* relations as primitives, which can supposedly hold between physical entities.

Fig. 1

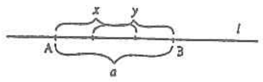

Fig. 2

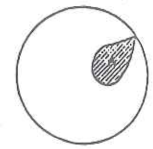

Fig. 3

Looking at *Fig. 1*, one sees that there is an intuitive sense in which the shaded area *A* is an *ending* part of the whole circled area, whereas the shaded area *B* is an *internal* part of it. Given that the circled area is a part of the plane on which the circle is placed, part *a* of the boundary line, as the outside limit of *A*, can be unequivocally defined as an *equivalence class* generated by *A*, namely, as the set of all parts of the outside area which are touched by *A*.

Let w, x, y, z range over proper parts of the plane on which the circled area is placed, let $x \subset y$ be read as 'x is a proper part of y', let $x \subseteq y$ be shorthand for $x = y \vee x \subset y$ and let xTy mean 'x touches y', where $\subset$ is supposed to be transitive and T to be symmetric.

In congruence with standard geometry, it should be supposed that $(x)(\exists y)y \subset x$, $(x)(\exists y)x \subset y$, $(x)(y)(xTy \Rightarrow \neg(\exists z)(z \subseteq x \wedge z \subseteq y))$ and $(x)(y)(\neg(\exists z)(z \subseteq x \wedge z \subseteq y) \Rightarrow xTy \vee (\exists w)(xTw \wedge yTw))$.

Notice that, in order to preclude the possibitity of 'holes' within the plane over whose parts our variables range, the ordinary touching relation should be supplemented by a contiguity relation as an ordered and asymmetric touching relation, say $T^{\rightarrow}$, so that for any chosen direction within the plane holds that $(x)(\exists y)xT^{\rightarrow}y$. As a consequence, the plane would be continuous and infinite in any direction and our variables would range over parts limited in any direction.

Now, let $x \subseteq_e y$ be read as 'x is an ending part of y' and be defined as follows:

$$x \subseteq_e y \rightleftharpoons x \subseteq y \wedge (\exists z)(xTz \wedge \neg(\exists w)(w \subseteq y \wedge w \subseteq z)).$$

A limit of *y* - such as segment *a* of the circle in *Fig. 1* - can be defined as equivalence class [*x*] generated by *x*, namely:

$$[x] = \{x | x \subseteq y \wedge xTz \wedge \neg(\exists w)(w \subseteq y \wedge w \subseteq z)\}.$$

It is easy to see that the last definition is applicable whenever *n*-dimensional limits are to be defined *via* (*n*+1)-dimensional entities. For instance, points *A* and *B* at *Fig. 2* can be defined as equivalence classes [*x*] and [*y*] of all parts of line *l* which lie outside segment *a* and which touch a left endsegment *x* and a right endsegment *y* respectively.

Although limits are not supposed to be parts of entities whose limits they are, they themselves can have parts, which holds for surfaces as limits of bodies just as for lines as limits of surfaces. Only points as limits supposedly have no parts.

Now, an entity is *n*-dimensional, for $n>0$, if and only if it has (*n*-1)-dimensional limits and no limits whose dimension is greater than *n*-1.

It is to be noticed that it is not said that *n*-dimensional entities can have only (*n*-1)-dimensional limits. It is reasonable and congruent with standard geometry to accept that 'being a limit' is transitive, so that limits of a limit of an entity are also limits of the entity itself. An equivalence class generated by an ending part of an *n*-dimensional entity can define a limit whose dimension is *n-k*, for $k>1$, as is the case with ending part *A* at *Fig. 3*.

It seems now that we can say - in respect of the first critical remark at least - that *there can be* geometric entities in the physical world. It is enough to let individual variables range over proper parts of an *n*-dimensional physical space, where *n* is standardly equal to 3, and obtain all geometric entities as actually or possibly existent (*n-k*)-dimensional limits ($0<k\geq n$). It seems resonable to say that geometric entities introduced in such a way are *actually existent in the physical world* if and only if they are limits between *physically distinguished*, i.e. heterogeneous, parts of the physical space, and *only possibly existent in the physical world* if and only if they are limits between *physically not distinguished* but yet *physically distinguishable* parts, where parts distinguished and distinguishable are *any* ending part generating the equivalence class associated with a given limit and *any* member of the equivalence class.

True, one could still argue that it should rather be said that there are not geometric entities in the physical world, if they are supposedly *completely reduced* to *non*-geometric entities. This is, however, a tricky remark. One can say equally well, as we have done, that there *are* geometric entities in the physical world *just because* they are reducible to *physical* ones.

Of course, the question of whether *only those* geometric entities should be said to *exist* which are reducible to physical ones is itself at issue and can be traced back to the dispute between Plato and Aristotle. But, in order to raise the central question we are going to deal with, it is sufficient to accept that geometric entities can be said to exist *in the physical world* only by *supervening* on the physical in the sense just explained.

2. It is said above that those and *only those* geometric entities should be said to be possibly existent in the physical world which are supposed to be actualizable as boundaries between physically distinguished parts of the physical space. Now, it can be remarked that it is hard to believe that a perfect square, for instance, could ever be actualized in such a way.

However inessential this remark seems to be, it can be of use for the clarification of the main concern of the paper. Namely, however *highly improbable* it can be that there are perfect squares, circles, etc., in the physical world, it is *not impossible* that such entities exist in the physical world. We shall not deal with *probabilities*, but with *possibilities*. Quite concretely, we are going to deal with the question *which* of the *mathematically definable* geometric entities are *possibly existent in the physical world.*

3. The third remark is similar to the previous one. It concerns limitations imposed by our *current knowledge* about what is *physically* possible. For instance - and this will turn out to be an important point in the following discussion - if a curve is to be drawn by an endlessly oscillating motion, such that the period of the function defining the motion becomes always smaller and smaller, the motion can be said to become physically impossible after a certain point. However, we shall ignore *such* limitations. We shall *extrapolate* physical possibilities to their *imaginable maxima* and take into account only *those* limitations - if there are any - which are imposed by *a priori* grounds concerning the *very relation* between the *mathematically definable* and the *physically possible*. The examination of such limitations can be said to be one of the two main concerns of the paper. The second one is *complementary* and relates to the mathematical way of proceeding which conceals the difference between those geometric entities which are possibly existent in the physical world and some other, very similar but yet only *abstract* objects.

We can now turn to our central question.

2. Remarkable curves

Let us define a function *f*(*x*) piecewise in the following way:

$$f(x) = \begin{cases} x sin 1/x & \text{for } -2/\pi \leq x < 0 \text{ and } 0 < x \leq 2/\pi, \\ 0 & \text{for } x = 0. \end{cases}$$

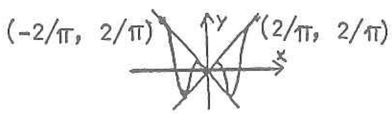

Fig. 4

The graph of *f*(*x*) (see *Fig. 4*) is a wave curve continuous on the interval [-2/π, 2/π], having, at the same time, no tangent at the point (0, 0).

The given curve is said to be *remarkable* just because it is continuous on an interval within which there is a point at which it has no tangent.

In order to apprehend more clearly the *mathematical way of proceeding* underlying the *non-coextensiveness* between *being continuous* and *being everywhere differentiable*, it will be of use to see how a curve *continuous everywhere* but *differentiable nowhere* is to be obtained by a *purely geometric construction.*

Fig. 5

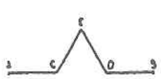

Fig. 6

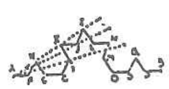

Fig. 7

Let us start with the segment AB represented in *Fig. 5* and let us delete the middle third so that the endpoints C and D are left standing. Let us replace the gap between C and D by a peak, by constructing the two sides of an equilateral triangle over the deleted third, obtaining in such a manner the broken line ACEDB represented in *Fig* 6. Let us proceed by doing the same with AC, CE, ED, and DB as we have done with AB, obtaining in such a way the broken line AFHGCIKJELNMDOQPB represented in *Fig. 7*. Let us now do the same with segments AF, FH, HG, GC, CI, IK, KJ, JE, EL, LN, NM, MD, DO, OQ, QP and PB as has been done with the original segment AB, and let us imagine that the process of replacement of the segments obtained in such a way has been analogously carrying on endlessly. After the completion of the infinite process of replacement, allegedly possible, we shall finally obtain a continuous curve as a limiting figure which has a tangent at no point.

The figure obtained as a limiting figure is *continuous* because the size of the piled-up peaks always becomes smaller and smaller. It has, at the same time, *no tangent* because the path of a point moving from A, let us say, along the approximating poligon last drawn, will rise many times to a summit (as in E and H) and drop many times to the original segment AB (as in G and F), so that, if the process of infinite replacement is supposed to be completed, the secants passing through A will move up and down infinitely often, moving back and forth between 60° and 0° without approaching a definite limiting position. What holds for A holds also for any point in the neighbourhood of A as well as for any point on the limiting figure.

The given construction of an everywhere continuous but nowhere differentiable curve has been invented by von Koch. However, in view of the historical fact that Weierstrass was the first who defined a continuous but completely non-differentiable function[2] - the function $f(x) = \sum_{n=0}^{\infty} b^n cos(a^n x)$, where a is an odd integer and b a positive constant less than 1 such that $ab < 1 + 3\pi/2$ - we shall call all curves without tangents *Weierstrassean curves.*

Once the Weierstrassean curves had entered the historical scene, mathematicians were quick to construct various curves which are remarkable not only as being continuous and, at the same time, non-differentiable, but also as exibiting some other astonishing properties.

Let us look, for instance, at what is represented in *Fig. 8*.

Fig. 8

One would certainly say, without hesitation, that it is a square, i.e., a *two*-dimensional figure. But, as Peano showed 1890, it can be viewed as a curve as well, i.e., as an *one-dimensional* figure.

Let us start dividing a unit square into $(2^1)^2$, $(2^2)^2$,..., $(2^n)^2$,... squares of equal size, each time connecting the centre points of the squares obtained through a division by a single broken line, as shown in *Fig. 9, 10* and *11* for n=1, n=3 and n=6 respectively. Supposing that the infinite process has been not only carrying on endlessly but also completed, the resulting figure can be shown[3] to be a Weierstrassean curve passing just once through every point of the original unit square.

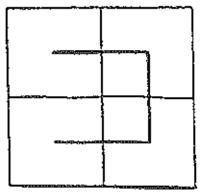

Fig. 9

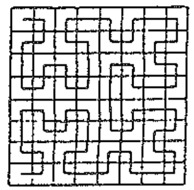

Fig. 10

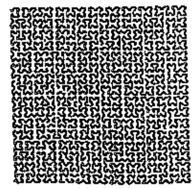

Fig. 11

Now, what can it mean that each point of a continuous line coincides with just one point of a square? Does it mean perhaps that there is no clear distinction between entities of different dimensions any more?

Fortunately, Brouwer[4] was quick to show us, 1911, that a clear differentiation between entities of different dimensions can still be established through the fact that a portion of a k-dimensional continuum can *only* be put into a one-to-one correspondence to another portion of a continuum if the latter one is *also* k-dimensional, and *never* to a portion of an m-dimensional continuum if $m \neq k$.

Let us illustrate Brouwer's result with a comparison between Cantor's correspondence contained in his famous proof that the cardinal number of the set of points of the unit segment is the same as the cardinal number of the set of points of the unit square and a correspondence between the set of points of the unit segment and the set of points of Peano's curve.

Let the left endpoint of a segment of a line correspond to zero, written as 0.000..., and the right endpoint correspond to 1, written as 0.999... . Let, according to the so-called Dedekind-Cantor axiom, every point lying between the two points correspond to just one real number, written as $0.a_1a_2a_3...$, where at least one digit in any decimal expansion differs from zero and 9. According to the same axiom, each point of a square as a *two*-dimensional entity can be brought into one-to-one correspondence with just one *ordered pair* of real numbers, each of the numbers written

as above. Now, there is one-to-one mapping of the set of points of the segments *onto* the set of points of the square, because each number written as $0.a_1a_2a_3...$ corresponds to just one number written as $(0.a_1a_3...a_{2k-1}..., 0.a_2a_4...a_{2k}...)$ (k=1, 2,...), and *vice versa*. However, the correspondence is *not* continuous, which is easy to see by comparing neighbouring points of the segment with the corresponding image points of the square. Contrary to this case, the set of points of the unut segment *can* be put into one-to-one *and* continuous correspondence with the set of points of Peano's curve and *vice versa*, since this holds for any two limited lines. According to Brouwer's proof, such an outcome is not accidental.

Wittgenstein[5] was very much impressed by the fact that *one and the same* figure can be *seen* in two *different* and *incompatible* ways, as a rabbit's head and as a duck's head, for instance (see *Fig. 12*). He should have been, however, much more impressed by the fact that two figures which coincide completely can be *analytically conceived* as figures of *different dimensions*, which, according to Brouwer's proof, are *irreducibly different entities* and not just different *aspects* of one and the same figure.

Fig. 12

Let us turn, finally, to the Weierstrassean curve invented by Sierpinski, which exibits an additional curious property.

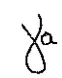

Fig. 13

A point on a curve is said to be *branch point* if and only if the boundary of any of its arbitrarily small neighbourhoods has more than two points in common with the curve, as point *a* in *Fig. 13*.

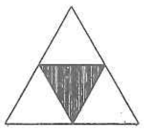

Fig. 14

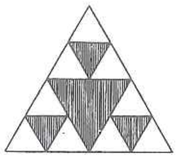

Fig. 15

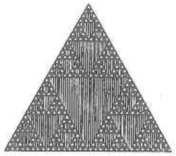

Fig. 16

Let us suppose that an equilateral triangle has been inscribed within a given equilateral triangle and the interior of the inscribed triangle erased - as represented by the shaded area in *Fig. 14* - its sides having been left standing. Let us suppose that the same has been done with each of the three triangles remaining, then also with each of the nine triangles obtained, and so on endlessly with each of 3^n triangles (n=1, 2,...), as shown in *Fig. 15* and *16* for n=2 and n=5 respectively. The points of the original triangle that survive the infinitely numerous erasures can be shown to form a curve *all* of whose points, with the exception of the vertex points of the original triangle, are *branch points*.

3. Remarkable curves and the physical world

Is it *possible* that there are *remarkable curves* in the physical world, given that there *can be* geometric objects in the physical world in the sense explained in Section I?

It has turned out that, from a mathematical standpoint, *one-dimensionality* together with *continuity* is the only *necessary* and *sufficient* condition for being a curve (if straight and broken lines are understood as special cases of curves). So, our question turns out to be about whether to be continuous and one-dimensional is *also* sufficient for being possibly existent in the physical world.

Let me start by citing two characteristic opinions about remarkable curves which seem to be prevailing in the whole twentieth century philosophy of mathematics.

Hans Hahn, speaking about the "crisis in intuition", states that "again and again we have found that in geometric questions, and indeed in simple and elementary geometric questions, intuition is a wholly unreliable

guide" and that "it is impossible to permit so unreliable an aid to serve as the starting point or basis of mathematical discipline".[6] But, to be sure, Hahn doesn't speak about mathematics as a *self-contained* and *isolated* discipline only. He compares the situation which has arisen through the invention of remarkable curves with the situation when "the theory that the earth is a sphere was also once an affront to intuition",[7] implying clearly that such an achievement in mathematics can have something to do with what is possibly existent.

Partly contrary to the opinion of Hahn, Karl Menger states that "if a word which already has a meaning attached to it in daily life is to be precisely defined in science, there is no reason for setting it in contradiction to the daily usage of the word, that is, for excluding from the concept things generally designated by the word under consideration, or including in it things generally not designated by it". He states that "hence, a *formal requirement* for the rigorous definition of a word appearing in the colloquial language is the following: it should *make precise and complement the usage of the word* which is vague and incomplete in border cases *without contradicting the same*".[8] It is important to note that Friedrich Waismann claims, arguing in favour of Menger's view, that it is exactly the conception according to which the line is a *boundary between two coloured surfaces* which *doesn't* contradict the *mathematical definition* following from the generalization implied by the discovery of remarkable curves.[9]

The point *common* to both these views is that *what* should finally be regarded as a curve and *whether* and *how* lines are at all to be distinguished from other geometric entities is to be decided within mathematics itself on the basis of its purely analytic way of proceeding and independently of any intuition or visualization. So, for instance, however one may be shocked by the statement that what is represented in *Fig. 8* can be a curve, it *can* be a curve *because* it is *analytically definable* as a curve, and however he may be shocked again by the statement that, in spite of that, a curve can never be said to be a square and *vice versa*, it must be accepted that it *is* so *due to* the clear and non-exceptional *analytic differentiation* between one- and two-dimensional entities.

The *difference* between the two views lies in the fact that, according to Hahn, the concept of curve should rather be said to be *re-defined* on the basis of the historical development of mathematics, whereas Menger and Waismann hold instead that what is done by Weierstrass, von Koch, Peano, Brouwer and others represents actually a *discovery* of new, astonishing properties of entities which still can be said to be curves in the *same sense* in which curves were conceived *before* this discovery.

To put the last point quite strictly, Menger and Waismann hold that the *necessary* and *sufficient* condition for being a curve has *always* been thought to be *one-dimensionality* together with *continuity* - what has turned out to remain *unchanged* in the modern analytic definition of curve - only that, once, mathematicians thought *wrongly* that the two properties imply *by themselves* that there are no curves without tangents, curves coinciding with all points of two-dimensional figures, and so on. It was allegedly *not a wrong conception* of curve which prevailed in the mathematics of those days, but rather the *'paradisical' state of ignorance*, as Waismann puts it by citing Felix Klein, "in which one did not distinguish in the case of a continuous function between good and evil, differentiable and non-differentiable".[10]

Now, I want to argue that the two views - that of Klein, Menger and Waismann and that of Hahn - are both false or, at least, misleading, because an *essential* aspect, concerning just the *relation* between mathematics and the physical world, is either not taken into consideration at all or it is oversimplified at least.

Let us start with the simplest case, with the wave function defined at the beginning of Section II. It is clear that the graph of the function can be nothing but a limited, continuous and one-dimensional object, i.e., a curve, according to the modern analytic definition. So far, so good. The question is, however, whether such an object can exist in the physical world, given that geometric entities supervene on physical entities in the manner in which we have introduced them in Section I. When dealing with this question, we are asking for *necessary* and *sufficient* conditions for *being a curve possibly existent in the physical world* and *not just* for *being an analytically definable curve*. Starting *without* prejudices of *any* kind, one *should* admit that there is *no* reason to believe that 'being analytically definable' *implies automatically* 'being possibly existent in the physical world'.

True, according to a well-known theorem from model theory, if a set of sentences is *consistent*, it has a *model*. But, it is not said that such a set must have a model within the *physical* world. *Restrictions* concerning the *rules* of the *language game* related to a given relational-operational structure must *also* be taken into account. I can understand, for instance, what '-500 dollars' means, when I am being informed about the state of my bank account. Negative numbers can be attached to horses in

a similar way, in order to express the fact, say, that I owe you two horses which I have to bring you back. But it makes *no sense* to speak about negative two horses existing in your stable, or anywhere. Any number of existing horses must be non-negative.

Now, both Hahn and Waismann *accepted* Jordan's (and actually Xenocrates') definition of a (limited) curve as a geometric figure which can be generated by a point running along in continuous motion.

Let us suppose that a god, say Zeus, has constructed a device, a *curve drawing machine*, consisting of two - quite imaginable - unlimitedly elastic cubes of the same size, heterogeneous amongst themselves and fitting perfectly snugly against each other, which, when being moved along a perfect plane, produce a one-dimensional, i.e. geometric, line as a limit between the two physically distinguished two-dimensional traces produced by their bases fitting also perfectly snugly against the plane.

Notice that, by introducing such a curve drawing machine, we ignore completely and deliberately the second remark dispelled in Section I, and we will also ignore the third one. Notice also that any curve produced by any motion of the curve drawing machine along a perfect plane fits Waismann's first definition of curve as a boundary between two surfaces heterogeneous amongst themselves, which is allegedly congruent with the analytic definition.

Now, it is easy to see that Zeus will have no problems with drawing any segments of *ordinary* curves, such as circles, parabolas, and so on. How is he to proceed, however, in trying to continue to draw the graph of the wave function under consideration, after he has supposedly reached the point (0, 0)? He *mustn't* start moving the curve drawing machine along the *x*-axis. He *mustn't* take *any* direction entering the *First* Quadrant. He *mustn't* take *any* direction entering the *Fourth* Quadrant. But, at the same time, he *should* enter the part of the whole plane consisting of just the First and the Fourth Quadrant. Therefore, he *cannot* do what he *should* do.

I can't see *any* manoeuvre which could help Zeus in the given situation. It is up to the reader to try to *find* and to *support* a possibly different answer.

Just as it is impossible for Zeus to *start* drawing the graph of the given function by beginning at the point (0, 0), it is impossible for him to *reach* the same point from the left. Namely, given that traces of the two bases of the curve drawing machine are *not* to be obtained by the bases reaching the point (0, 0) *from* Quadrant 2 or *from* Quadrant 3 (including the negative part of the x-axis), and that the point (0, 0) should still be reached by a motion along the part of the whole plane consisting of just Quadrant 2 and Quadrant 3, the point could be reached only from *nowhere*.

Therefore, the graph of the given wave function *can't* be drawn by Zeus' curve drawing machine *at all*, *in spite* of the fact that it is *one-dimensional* and *continuous*.

If Zeus can't draw the graph of the given wave function, he can't draw, *a fortiori*, von Koch's curve. The three definitions of curve - defining it as 1. a boundary between two parts of a plane heterogeneous amongst themselves, 2. a geometric figure generated by a point running along in continuous motion, 3. an one-dimensional and continuous object defined analytically - are *not coextensive*, contrary to the alleged continuity in the development of mathematics claimed, in relation to this point at least, by Menger and Waismann.

More than that, however, the first and the third definition *alone* are not coextensive, independently of the second one.

Let us raise the question, for instance, whether that which is represented in *Fig. 8* can be a curve according to the first definition, which is actually coextensive with our definition introduced in Section I, of a curve possibly existent in the physical world (if entities of dimensions greater than two are not taken into account). Now, although lines represented in *Fig.* 9, *10* and *11*, as well as *any* line involved in the infinite construction aproximating Peano's curve, can be said to be curves according to the definition under consideration, this doesn't hold for Peano's curve *itself*, simply because, after the alleged completion of the infinite process, there could be *no place any more*, within the original square, for *any* heterogeneity. It is not difficult to see that nothing essentially would be changed if we were allowed to take into account the entities of higher dimensions too, given that it is supposed - as it should be, in order to be in a position to let our individual variables range over physical entities of the highest dimension - that there *are* entities of the *highest* dimension (the number of dimensions being traditionally equal to three).

Similarly to the case discussed, the area of the original triangle within which Sierpinski's curve is to be obtained leaves *no free place*, after an infinite number of erasures, for *any* heterogeneity.

It can be said that, while the square associated with the corresponding Peano's curve should become *completely 'black'*, the triangle associated with the corresponding Sierpinski's curve should become *completely*

'white', which implies the *impossibility* of the existence of both the curves in the physical world.

Let us now turn to Hahn's view. Contrary to Menger and Waismann, Hahn *recognized* the *discontinuity* in the development of mathematical thinking affecting the topic discussed, which fits the non-coextensiveness between the three cited definitions of 'curve'. But, the real founder of the Vienna Circle and friend of Wittgenstein's *Tractatus* rejected decisively the attempt to examine *philosophically* the possibility of the existence of remarkable curves *in view* of the discrepancies between the three definitions cited. The only question related to the existence of geometric objects which he was willing to allow to be dealt with *a priori* is the question of *internal* and *external* existence in Hilbert's sense. The question of internal existence simply amounts to proving within an axiom system a statement of the form $(\exists x)A$. The question of external existence is whether the system has an interpretation, and the sufficient condition for that is that the system is consistent. The questions of internal and external existence are to be answered by *logicians* and *mathematicians* only. *Anything else* which can be asked about the existence, or possible existence, of mathematical structures concerns the question of how much they are *well suited* for the interpretation of the observational data thus far accumulated, and such a question can be answered, according to Hahn, only indirectly and *a posteriori* by *physicists*.

I don't think that philosophers should be that modest. More than that, I hold that they *mustn't* be that modest. There should be somebody to say that the ontology of points, lines and surfaces supervening on the physical is *incompatible* with the possible existence of remarkable curves in the physical world for *a priori* reasons.

However true it can be that mathematicians have always *believed* that the *necessary* and *sufficient* condition for an entity being a curve is that it is *one-dimensional* and *continuous*, that which they once actually *thought* to be a curve and which is expressible through the first two definitions cited is *not* the same kind of entity as that which is implied to be a curve by the contemporary analytic definition, because the former definitions *imply* the *possibility* of existence in the physical world, and the latter one *doesn't*.

We have established that for being a curve possibly existent in the physical world it is not sufficient to be one-dimensional and continuous. Some *additional* properties, which remarkable curves are *deprived* of, have turned out to be *also necessary*.

The remarkability of remarkable curves is not only *psychological*. It has also something to do with the *physical*.

4. Remarkable curves and the mathematical treatment of infinity

If one sympathizes with the *arguments* from Section III, supporting the impossibility of the existence of remarkable curves in the physical world, he can still be reluctant to accept the *conclusion* before being shown, in addition, by looking at the mathematical way of proceeding, *why* the equation of a circle, for instance, defines a *possibly existent* and the equation of a remarkable curve an *impossibly existent* object. One's illusion can't be dispelled by his being confronted with the consequences only. A *source* of the illusion should also be detected and the illusion *explained*.

If a function is defined, we know absolutely precisely *which* value of $f(x)$ is to correspond to *any* given value of x. So, *how* is it possible that the graph of a continuous Weierstrassean function is an impossible object in the physical world *if* points are *allowed* to exist in the physical world and *each* of the points of the graph can be found precisely?

We seem to be confronted with a *paradox* - or an *antinomy* in the Kantean sense - where different arguments from the pile lead to mutually contradictory conclusions.

There is one thing related to the remarkable curves discussed - and actually to any Weierstrassean curve - characteristically missing from ordinary curves, which hints at a possibly essential difference between the former and the latter: all remarkable curves - or at least all remarkable points on them - are supposed to be obtained through an *infinite process* allegedly *completable*.

I don't want to repeat various arguments, I dealt with elsewhere,[11] directed against the possibility of completion of an infinite process. For figuring out the essential difference between ordinary and remarkable curves, concealed by, or deliberately ignored in, the mathematical treatment of infinity, it will suffice to point out what happens with ordinary segments of ordinary lines when they are allowed to be viewed from the infinitist standpoint.

In order to proceed in accordance with the manner in which lines and points are understood in Section I, we shall not treat lines as consisting of points as *entities in themselves*, but we can still try to treat a segment of a line as *consisting* of an *infinite number* of *non-overlapping subsegments* whose limits are allowed to coincide (in accordance with the fact that limits are *not* understood as parts). For instance, a unit segment can be attempted to be understood as consisting of its half, an abutting quarter, an abutting eighth, and so on.

Now, the unit segment under consideration can't be a segment closed on both sides, and, as a (half)-open segment, it is an object which is *not* possibly existent in the physical world.

That the segment can't be closed on both sides follows from mathematics itself. That a (half)-open segment is an impossible object in the physical world follows from the fact that there can be no limited physical bodies without limits belonging to them. In order to draw a half-open sgment, Zeus would need a curve drawing machine consisting of cubes such that each of them *doesn't have* an endsurface.

Consequently, a physically real segment *can't* be viewed as consisting of an infinite number of *actual* subsegments,[12] although *any* of its subsegments taken *per se* is supposed to be *actualizable*. True, *by drawing a real segment* Zeus will also draw an *infinite* number of subsegments *definable mathematically*, but he can't obtain a real segment in the opposite way, by drawing each of the mathematically definable segments *separately*.

The last point explains why it is impossible for a limited physical body to consist of an infinite number of parts physically distinguished amongst themselves.[13] But it can explain *also* why there can be circles but no remarkable curves in the physical world. Namely, contrary to an ordinary (closed) segment of an ordinary line, remarkable curves are *infinitely heterogeneous at the second-order level*. For instance, the wave curve we have dealt with should have an infinite number of *turning points*. It is *just* this *infinite heterogeneity* which makes the corresponding wave function *non-differentiable* at the point (0, 0).

It is of *crucial* importance to notice that *nothing* in the above explanation depends on limitations which *physicists* could impose from *their* standpoint. Nothing depends on too small smallness or too great greatness. Zeus is *allowed* to carry the infinite process further and further on, when he tries to obtain a remarkable curve. But he can't succeed all the same.

The reason for Zeus' failing to succeed depends *only* on the *logic of infinity* applied to the *physical world*. If one starts by letting individual variables range over entities of the highest dimension, where the number of dimensions has to be finite, he *can't* get an infinite heterogeneity at *any* level.

There is *no greatest* number of waves of the wave curve actualizable in a *possible* case, but the number of them must be finite in *any* possible case. Mathematicians may speak correctly of an infinite number of waves *not* for the reason that they speak of no *actual* cases, *but* for the reason that they speak about no *particular* case, be it actual or possible, so that the number of waves is *not fixed* and can be said, consequently, to be greater than in any particular case. This is the *only* reason why *some* mathematically definable geometric objects are *abstract* entities.

The argument that for the possible existence of a continuous one-dimensional entity it is sufficient that it is analytically definable by a function, because we know precisely which value of $f(x)$ is to correspond to any given value of x, ceases to be convincing when we remember that points should be points of one and the same *otherwise* actualizable line. Mathematicians ignore any 'otherwise', in accordance with their way of proceeding, and take all actualizable members of an infinite series as *actualizable simultaneously*, which is not always the case, $(x)\alpha A$ not implying $\alpha(x)A$ ('for any x, it is possible that A' doesn't imply 'it is possible, for all x', that A'.

5. Concluding remark

Independently of how the reader will judge the value of the above arguments, he will hopefully accept, as a general lesson, that there are intriguing questions concerning the relation between mathematics and the physical world which can be dealt with in an *a priori* manner, not only by examining language games played by mathematicians and physicists, but also by using the *strong reductio ad absurdum* method,[14] which Aristotle called *dialectic*,[15] intended to test the plausibility of various views about the possible existence of mathematical entities. This is a *typically philosophical* job, not to be done by physicists *as* physicists and mathematicians *as* mathematicians. Physicists *qua* physicists would be too dogmatic and mathematicians *qua* mathematicians too liberal.

Of course, philosophers can do their job badly, and mathematicians and physicists are invited to help them. But, in doing that, they have to become philosophers. A musician can happen to cook better than a professional cook, but when he cooks, he is a cook.

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Notes

1 See Aristotle 1831, Physica 226 b 18 ff., Topica 141 b 15 ff.
2 See Weierstrass 1894-1927, II, p.71-74.
3 See Hahn 1921, p. 146 and Menger 1932, p. 10.
4 See Brouwer 1911, pp. 161-165.
5 See Wittgenstein 1980, p. 16.
6 Hahn 1980, p. 98.
7 Ibid. p. 100.
8 Menger 1928, p. 75.
9 Waismann 1966, p. 165.
10 Ibid. p. 159.
11 See, for instance, Arsenijević 1992, II.
12 Cf. Weyl 1949, p. 42.
13 Cf. Arsenijević 1989.
14 Cf. Ryle 1971.
15 Cf. Aristotle 1831, Topica 101 a 34 and 101 b 1.