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Logic, Mathematics and Philosophy

The main concern of Michael Hallett's illuminating paper about the relation between logic, mathematics and philosophy is to show that Hilbert's attempt to address the question of 'external existence' through mathematics and logic alone, although brilliant, is, in Hallett's words, a "significant failure", largely because the route through consistency does not after all make the issue of existence any more accessible.

The question of *external* existence related to an axiom system, in contrast to the question of *internal* existence, which simply amounts to proving within the system a statement of the form $\exists x A$, is whether the system actually has an interpretation. The so-called categoricity demand is then the demand that the axioms *characterize* (at least up to isomorphism) the objects of the investigation, or, as Bernays once put it, that the axoms assert "the possible existence of a structure". For instance, an axiom system for geometry implicitly defines points, lines and planes, so that if there are three systems of objects mutually related so that the axioms are fulfilled (i.e., that under a suitable correspondence between names and the objects and relations the axioms become true assertions), then all the theorems of the system hold also for these objects and relations, the *necessary* and *sufficient* condition being, according to Hilbert, that the system under consideration is *consistent.*[1]

Now, Hallett's critique of such an approach is not primarily directed against the very idea of treating the objects and relational structures as implicitly defined through systems of axioms,

[1] See Hallett's paper in this volume, especially the passages from Bernays cited at the beginning of Section *1*.

but rather against its applicability in many important cases. Namely, as Hallett puts it in a preliminary version of his paper,

> "it was surely of the essence of Hilbert's approach to existence to show that the consistency problem (and thus the problem of external existence) is just of [an] elementary nature, for part of the point is not only to make existence a purely mathematical question, but also to put it beyond philosophical doubt".

And the consistency problem is not of an elementary nature (in many important cases), due to Gödel's Second Incompleteness Theorem.

In my *complementary* treatment of the relation between logic, mathematics and philosophy, I am going to deal with an apparently friendlier question, which Hallett has cited as an instance where mathematical analysis is "enormously fruitful". My point concerning that problem will be, however, the same as Hallett's point about the question of external existence: however fruitful the machinery of modern mathematical analysis is in giving the *precise formulation* of the question, it still doesn't make the answer any easier, since the problem remains *philosophical*, as it was before.

The problem to be considered is celebrated: is it possible (under some constraints to be examined) to complete a task consisting of an infinite number of physically distinguished and strictly discrete steps?[2]

Now, in contrast to the question of external existence, dealt with by Hallett, the consistency of standard mathematical analysis will be *presupposed*. In addition, I don't want to argue that the answer to the question whether it is possible to complete an infinite task or not can vary because there is *more than one* logic and *more than one* mathematical analysis (as there is). The point will be *neither* that different systems of logic and mathematics

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[2]Actually, the problem is the twentieth century version of the first two Zeno's kinematical paradoxes (*cf.* Weyl (1949)) p. 42.

imply different answers to the given question *nor* that logic and mathematics are not philosophically neutral. Logic will be supposed to be *classical* and mathematical analysis will be supposed to be *standard*. The point will be that the positive and the negative answer depend, in any case, on *extra* premisses too, the plausibility of which is to be judged only *philosophically*.

If a task consists of an infinite number of steps, then

$\forall m \exists n (n > m \wedge f_n)$,

where f_n denotes n-th step and $m = 1, 2, ...; n = 1, 2, ...$.

Each f_n is supposed to be performed in a time-period of positive duration, and if steps are to be strictly discrete, it should be supposed, in addition, that between each two f_n and f_{n+1} there is a time-interval of positive duration.

Now, it is necessary to introduce some general constraints, if the question is not to be trivialized from the very beginning, in one way or another.

First, it must be supposed that there is enough time for performing the process till any f_n, however great, so that

$\forall n \exists t f_n(t)$,

where $f_n(t)$ means that f_n is finished at or before t, t ranging over time-instants or over time-periods.

If there were no further restrictions concerning the performability of each of the steps, the infinite process could be *performing endlessly* according to $\forall n \exists t f_n(t)$, but it would not be necessarily *completable*, since

$\forall n \exists t f_n(t)$

doesn't imply

$\exists t \forall n f_n(t)$.

For instance, if each f_n lasts one second and t ranges over durationless instants in 1-1 correspondence with the real numbers or, starting from the beginning of the infinite process, over time-periods of the same positive duration in 1-1 correspondence with the natural numbers, the process is not completable, because

$\neg \exists t \forall n f_n(t)$.

So, if the completion of the infinite process is not to be precluded in a trivial way, additional constraints are needed. Quite generally, there are two possibilities: to allow t to range over transfinite numbers, too, or to let time-intervals in which steps of the infinite task are performed, as well as the pauses between them, become shorter and shorter, converging to zero.

The first possibility seems implausible, but if it is really implausible, it is so only due to the way in which time is normally conceived. It is to be noticed, already at this stage, that there can be *non-mathematical conceptual* reasons relevant for the problem under consideration.

The second possibility has been endorsed by all those who have argued in favour of the possibility of completing an infinite task[3].

Now, viewed from a strictly empirical standpoint, supported by Hilbert himself [4], one can be sceptical about the possibility that for some very great n, f_n can be performed in a corresponding, very short time-interval, but reasons supporting such a view, being *a posteriori* and trivializing the problem in a *non-inherent* manner, should be ignored *in the given context*. We will suppose that steps can be performed in an *arbitrarily* short time.

It seems now that the question about the possibility of completing an infinite task, defined with the use of standard analysis, must have a positive answer. The only thing still to be done, it seems, is to define f_n in a suitable way so that constraints concerning space, velocity and acceleration fit those concerning time in such a way that each f_n is feasible kinematically and dynamically. This is done by Grünbaum in many articles, and in his famous book *Modern Science and Zeno's Paradoxes.*[5] Grünbaum holds that the question under consideration is just an example of an allegedly typically philosophical problem which is not only *for-*

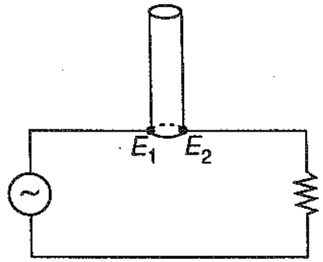
[3] See, for instance, Taylor (1951), Watling (1952), Maxwell and Feigl (1961), Grünbaum (1968a), pp. 80ff.

[4] See Hilbert and Bernays (1968) p. 16.

[5] See, besides Grünbaum (1968a), Grünbaum (1968b) and (1969).

mulable but also *resolvable* through *standard mathematical analysis.* In this last claim, Grünbaum goes too far, and I want in this paper to put forward some of reasons for doubt. (I was honoured to be in the situation, in Bielefeld, to cast doubts on this view in Prof. Grünbaum's presence.)

Let us consider a version of the famous Thomsonean Process[6] consisting of an infinite number of temporally separated upward and downward jabbing motions of the button of a lamp. Let the button base be initially at E_1E_2 (see the diagram), so that the circuit is closed.



After the button has been raised in 1/4 sec so that the base is 1/4 cm above E_1E_2, let it be at rest 1/2 sec. Then, after it has been pressed in (1/4 + 1/8) sec so that the base is 1/8 cm below E_1E_2, let it be at rest 1/4 sec. Then, after being raised again, this time in (1/8 + 1/16) sec, so that the base is 1/16 cm above E_1E_2, let it be at rest 1/8 sec, and let the process be continued endlessly so that vertical distances covered by uninterupted upward and downward motions and respective times in which they are performed are determined by the same sequence, whose first member is 1/4, the others being $1/2^2 + 1/2^3, 1/2^3 + 1/2^4, \ldots, 1/2^n + 1/2^{n+1}, \ldots$ $(n = 2, 3, \ldots)$. The time-intervals in which the button is at rest are determined by the geometric progression $1/2, 1/4, \ldots, 1/2^n, \ldots$ $(n = 1, 2, \ldots)$.

Now, the arithmetical sum of the members of each of the two sequences is equal to 1, which means that, if the infinite

[6] See Thomson (1954).

process could be completed, the total space interval traversed by the button would be 1 cm, whereas the total time would be 2 sec, the average velocity of each upward or downward motion being 1 cm/sec.

There may be specifically *dynamical* difficulties in effecting the infinitude of accelerations, because in ever shorter time-intervals of upward and downward motions the acceleration would increase (and decrease) boundlessly, and if the button should be definitely at rest after the two seconds have elapsed, the acceleration function would have an infinite discontinuity at the terminal instant of rest-and-zero-acceleration. These difficulties can be obviated by re-arranging the process according to Richard Friedberg's conditions[7], so that intermittent motions can proceed at suitably decreasing average velocities such that both successive peak velocities and accelerations during the decreasing subintervals converge to zero as we approach the terminal instant.

Given that the specifically dynamical difficulties can be removed through Friedberg's version of the Thomsonean process if the process is *otherwise* completable, we can ignore them and discuss the problem by using the original arrangement, because the two difficulties I will point out are of a quite different kind. The first difficulty is related to the fact that upward and downward motions are physically distinguished in view of their directions. The second one is of a more general nature and concerns any infinite task consisting of strictly separated steps.

If the infinite process could be completed, the button base should be at E_1E_2 after the two seconds have elapsed, because for any other position it holds that there is an instant within the open time-interval of the two seconds after which the base has never been raised (or pressed) so as to reach that position.

However, the final position of the button base could *not* be a direct consequence of a continuous downward or upward motion of positive duration, because for any such motion, however small,

[7] See Grünbaum (1969), pp. 213-214.

there is a time within the two seconds after which *only* shorter motions have been performed. If, due to some device, such as introduced by Allen Janis[8], the lamp were on or off depending on whether the button had arrived at E_1E_2 from above or from below, the lamp would be neither predictably on nor predictably off. "Having-come-from-above" and "having-come-from-below" being *only contrary* but *not contradictory* (because both can be false when attributed to a position of the button base), the answer to the question of whether the lamp would be on or off depends on what the reaction of the device to "having-come-neither-from-above-nor-from-below" is supposed to be.

Now, the difficulty is just that "having-come-from-above" and "having-come-from-below" should be only contrary but not contradictory in spite of the fact that the process consists *only* of strictly separated upward and downward motions of positive duration and that for *each* of these motions holds that position E_1E_2 can be arrived at *only* from above or below.

The most important thing to be noticed is that, once the difficulty has been formulated, the answer, that "having-come-from-above" and "having-come-from-below" are not contradictory *because* there is *no last* downward or upward motion, is *begging the question*. Of course, *if* the infinite process *were* completable, the final position *wouldn't* be reached either from above or from below, *because* there is no last downward or upward motion; but it is just *this implication* which raises the difficulty.

Let us now turn to the second difficulty. The total space-interval to be traversed by the button is 1 cm and this should be traversed in two seconds. If we put aside the first difficulty mentioned, then, given that it is feasible kinematically to perform each upward or downward motion, the task seems to be as performable as any other task which involves covering 1 cm in two seconds.

[8] See Grünbaum (1968a) p. 101.

However, there are perhaps essential differences between traversing a distance *legato* and traversing it *staccato*, even if there is a sense in which it can be said that an infinite task has been performed by a *legato* motion. Namely, it can be said that in the *legato* case an infinite number of distances is covered *as a consequence of a single act* (the *legato* motion), whereas in the *staccato* case the final position should have been reached *as a consequence of an infinite number of acts.*[9]

The problem of the infinite *staccato* task consists in the fact that after *any* of the strictly discrete motions only a finite number of steps have been performed, and it is not at all clear how the situation could be changed by performing the task *step-by-step*, because "finitude of the performed motions" is something which is being recursively preserved *step-by-step*.

To say that the situation changes when *all* the steps have been performed is again begging the question.[10] Of course, *if* the button *could* reach the final position in a *staccato* way, an infinite number of steps *would have been* performed.

The infinitists would certainly say — and this is now the central point — that all requirements for further explanations are simply *misplaced*, if there is *no formal inconsistency* between the implications of the statement that the infinite process under consideration is completable. We are *simply asked to accept* that the button base can reach its final position by coming neither from above nor from below (actually from nowhere), and that the ad-

[9] "[...]if the segment of length 1 really consists of infinitely many subsegments of lengths 1/2, 1/4, 1/8,..., as of 'chopped off' wholes, then it is incompatible with the character of the infinity as the 'incompletable' that Achilles should have been able to traverse them all" (Weyl (1949) p. 42). However, the truth of the *antecedent* of this conditional has been often denied or questioned since Aristotle (see, for instance, Aristotle (1831) 239 a 24ff., 185 b 9ff., 227 b 20ff., 236 b 7; see, also Hinton and Martin (1954), Arsenijević (1989), Arsenijević (1992)).

[10] TeHennepe argued that, in ordinary language *at least*, to *finish* an infinite process by carring it on endlessly is a straight contradiction (see TeHennepe (1963)).

dition of the two seconds terminal instant, at and after which no step is to be performed any more, implies that an infinite number of upward and downward motions has been performed. More than that, the infinitists might say that there is ***no way around this interpretation***, if we have accepted — as we really have — that

$$\forall n \exists t f_n(t),$$

because it holds in the given case, in contrast to the case from which we started, that

$$\exists t \forall n f_n(t),$$

the two seconds being a *finite* time.

But, is there really no other interpretation which would *consistently* maintain that also in the given case, in spite of the fact that the process is placed within two seconds, it is only ***endlessly performable*** but not *completable*?

Isn't it possible to turn the infinitist arguments on their heads[11] and use the difference between an *open* interval and the corresponding *closed* interval for stating that the infinite process under consideration *is performable* within the *open* interval of two seconds *without being completable* in the *closed* interval of the two seconds?

It is at least *not inconsistent* to say that the statement A, claiming that the Thomsonean process will continue endlessly within the open interval of two seconds, and the statement B, claiming that the two seconds will elapse, are *incompatible* even though $\Diamond A$ and $\Diamond B$, because

$$\Diamond A \wedge \Diamond B$$

doesn't imply

$$\Diamond(A \wedge B).$$

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[11] Cf., for instance, Benacerraf's arguments, directed against Thomson's anti-infinitism, in Benacerraf (1964).

The point of this anti-infinitist proposal[12] is neither that the Thomsonean process is not performable endlessly nor that the two seconds cannot elapse. The point is that the two are *not composable*, and that it is impossible — even in principle — to witness *both* the endless performance and the end of the two seconds. Even if a god were following the Thomsonean process carried on endlessly within two seconds, he would not be the witness of its end, because there is no end. True, in ordinary language "x will never happen" entails "x will not happen, even if we wait more than two seconds", but this entailment holds in *ordinary* language, which is a suitable language for *ordinary* experiences, in which we *don't* follow any Thomsonean processes.

So, even if both space-intervals and time-intervals in which the steps of an infinite *staccato* process are perfomed become smaller and smaller, converging to zero, $\forall n \Diamond f_n$ would be true, because there is *no greatest number* of steps which can be performed, but $\Diamond \forall n f_n$ would *still* be false[13], *an infinite number* of steps not being performable.

I don't want to issue a definite judgement about the two rival views discussed. The purpose of my paper has been only to show that there are *two consistent* views definable within *classical* logic and *standard* analysis under the *common* supposition that the infinite process is *performable*, the possibility of the *completion* of the process depending on the comparative plausibility of the implications of the two views, which is to be estimated *philosophically*. Unfortunately, or fortunately, the problem is still philosophical — as it always was.

One of many virtues of Prof. Scheibe's interdisciplinary work consists in clear separation and adequate treatment of the diff-

[12] Cf. Arsenijević (1988). This is not the answer of the other anti-infinitists (cf., for instance, Hilbert (1926), Wisdom (1952), Black (1951), Schwayder (1955)).

[13] This would satisfy Chihara (cf. Chihara (1965)).

erent aspects of a problem. This is the main methodological postulate which I have tried to follow in this paper.

References

Aristotle (1831), "Physica", in Immanuelis Bekkeri *Opera*, Academia Borussica, Berolini.

Arsenijević, M. (1988), "Solution of the *staccato* version of the Achilles Paradox", in *Contemporary Yugoslav Philosophy: The Analytic Approach* (A. Pavković ed.), Kluwer.

— (1989), "How many physically distinguished parts can a limited body contain?", *Analysis* **49**.

— (1992), "Eine aristotelische Logik der Intervalle, die Cantorsche Logik der Punkte und die physikalischen und kinematischen Prädikate" I and II, *Philosophia naturalis* **29**.

Benacerraf, P. (1964), "Tasks, super-tasks and the modern Eleatics", *Journal of Philosophy* **24**.

Black, M. (1951), "Achilles and the tortoise", *Analysis* **11**.

Chihara, C.S. (1965), "On the possibility of completing an infinite process", *Philosophical Review* **74**.

Grünbaum, A. (1968a), *Modern Science and Zeno's Paradoxes*, London.

— (1968b), "Are 'infinite machines' paradoxical?", *Science* **159**.

— (1969), "Can an infinitude of operations be performed in a finite time?", *British Journal for the Philosophy of Science* **20**.

Hilbert, D. (1926), "Über das Unendliche", *Mathematische Annalen* **95**.

Hilbert D. and Bernays P. (1968), *Grundlagen der Mathematik*, London.

Hinton J.M. and Martin C.B.(1954), "Achilles and the tortoise", *Analysis* **14**.

Maxwell G. and Feigl H. (1961), "Why ordinary language needs reforming", *Journal of Philosophy* **58**.

Schwayder, D.S. (1955), "Achilles unbound", *Journal of Philosophy* **52**.

Taylor, R. (1951), "Mr. Black on temporal paradoxes", *Analysis* **13**.

TeHennepe, E. (1963), "Language reform and philosophical imperialism: Another round with Zeno", *Analysis* **23**.

Thomson, J.F. (1954), "Tasks and super-tasks", *Analysis* **15**.

Watling, J. (1952), "The sum of an infinite series", *Analysis* **13**.

Weyl, H. (1949), *Philosophy of Mathematics and Natural Sciences*, Princeton.

Wisdom, J. (1952), "Achilles on physical racecourse", *Analysis* **12**.